Bending the rules of Geometry

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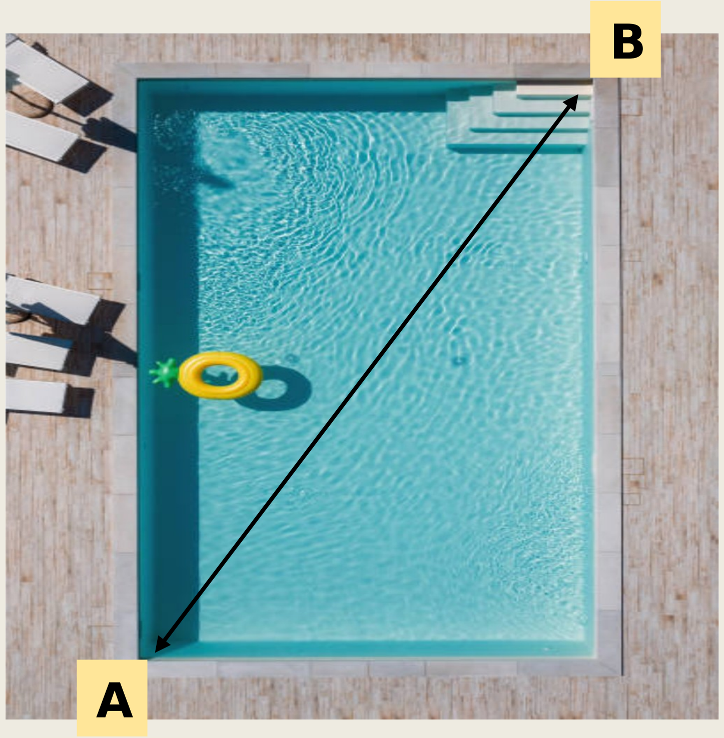
Yashodhan M.
Associate Software Engineer

Let us start with a simple question.

How many meters will a swimmer have to swim to get from point A to point B in the rectangular swimming pool as shown in the picture?



8 m

6m

Excellent!
Now let's see the second question.

The US State of Wyoming is rectangular with the dimensions:

600 km by 450 km (roughly).

What is the distance between point A and point B?

You can use the calculator.

It is not 750 km!

One more puzzle:

A man leaves his (only) home
and travels 10km towards the South.

Then he turns left and travels 10km to the East.

After that he turns left and travels 10km towards the North.

And he reaches his home!

Q: What is the colour of his pet bear?

Hint: the world map.


Hint: The Earth

The answer:
White

Explanation: He lives on the North Pole!

Hence, we are now starting to see that the rules of high-school geometry are not universally applicable.

But who made these rules?

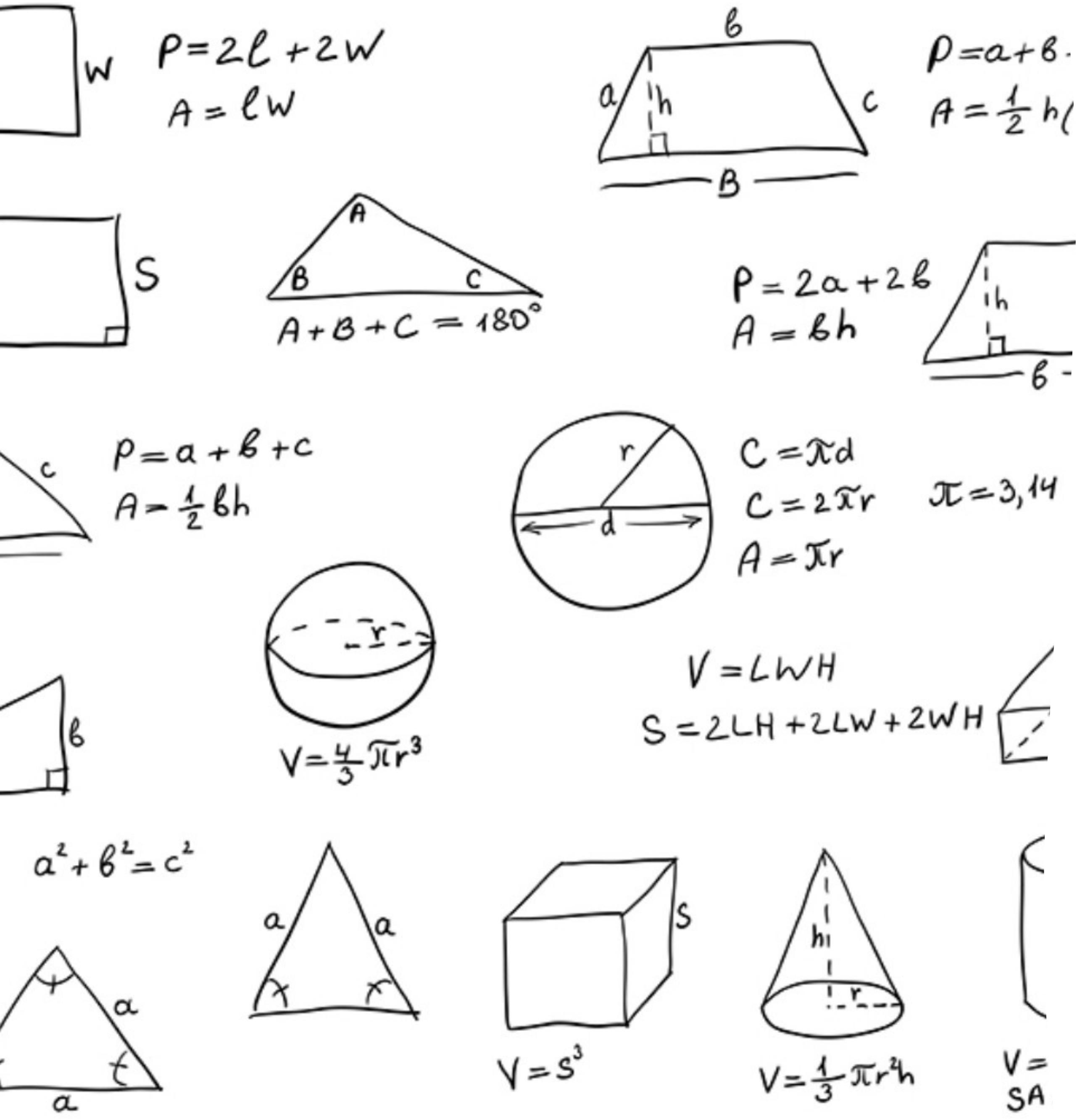
Euclid did!

Euclid (300 BC) was an ancient Greek mathematician active as a geometer and logician.

Considered the "father of geometry", he is chiefly known for the Elements treatise, which established the foundations of geometry that largely dominated the field until the early 19th century.

With Archimedes and Apollonius of Perga, Euclid is generally considered among the greatest mathematicians of antiquity, and one of the most influential in the history of mathematics.

(Copied from WikiPedia)



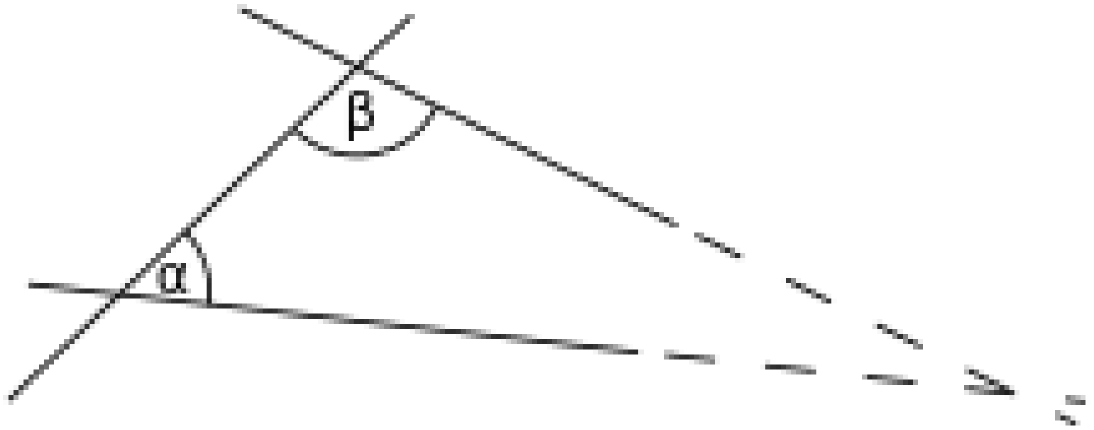
The following are the postulates of Euclidean Geometry (which remained undisputed for two thousand years):

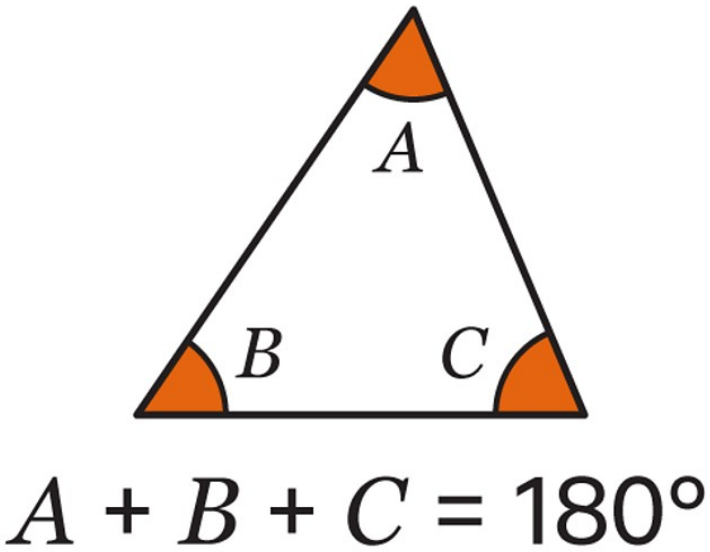
1. To draw a straight line from any point to any point.
2. To produce (extend) a finite straight line continuously in a straight line.
3. To describe a circle with any centre and distance (radius).
4. That all right angles are equal to one another.
5. [The parallel postulate]: That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than two right angles.

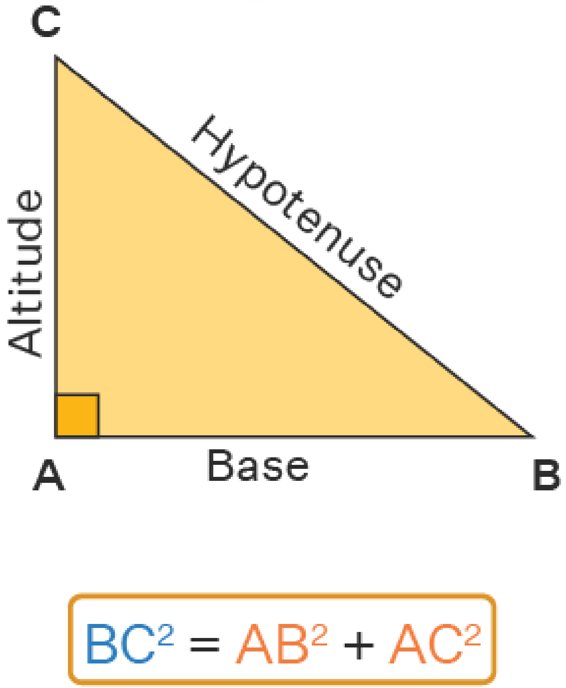
What is the fifth postulate?

- It essentially means that parallel lines never meet, or that one can only draw one parallel line passing through a given point to a given line.

- Using these five postulates we can prove every theorem in geometry.







Two important results of the five Euclidean postulates:

1. The sum of internal angles of a triangle is 180°.

2. The square of the hypotenuse of a right-angled triangle is the sum of squares of the sides.

Do these results apply to the problems we saw earlier?

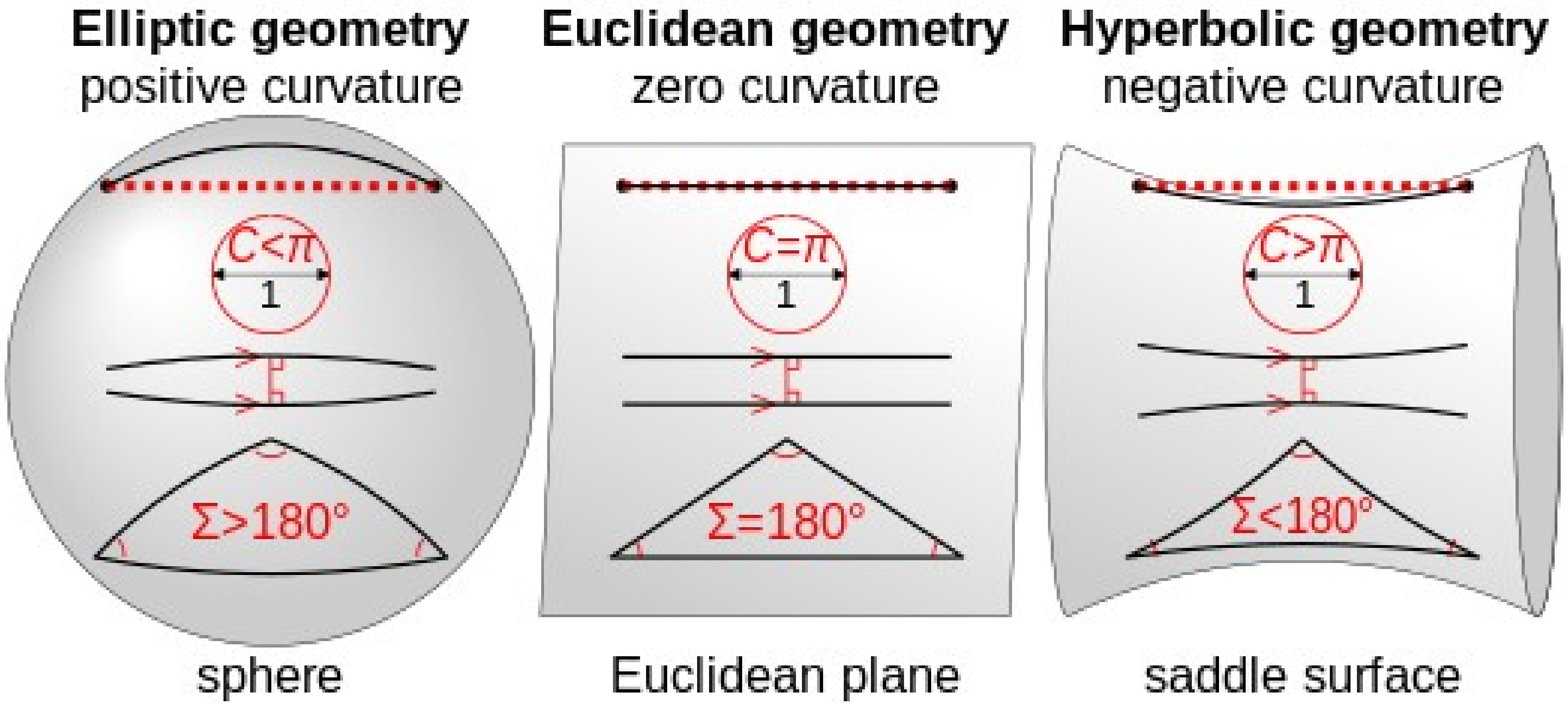
Here I have created a triangle between the Southern-most, the Eastern-most and the Western-most point.

If you add up all the internal angles of this triangle, you shall get 184° instead of 180°.

- As for the Pythagoras theorem, it is also not valid!
- Wyoming's width is roughly 588.2km and its height is 444.0km.
- The Pythagoras theorem then yields us the length of the diagonal as 737.0km
- The actual length is 722.2km!

- The results of Euclidean geometry are not applicable in this geometry.
- That means that one of the postulates of Euclidean geometry does not hold.
- It is the fifth postulate.

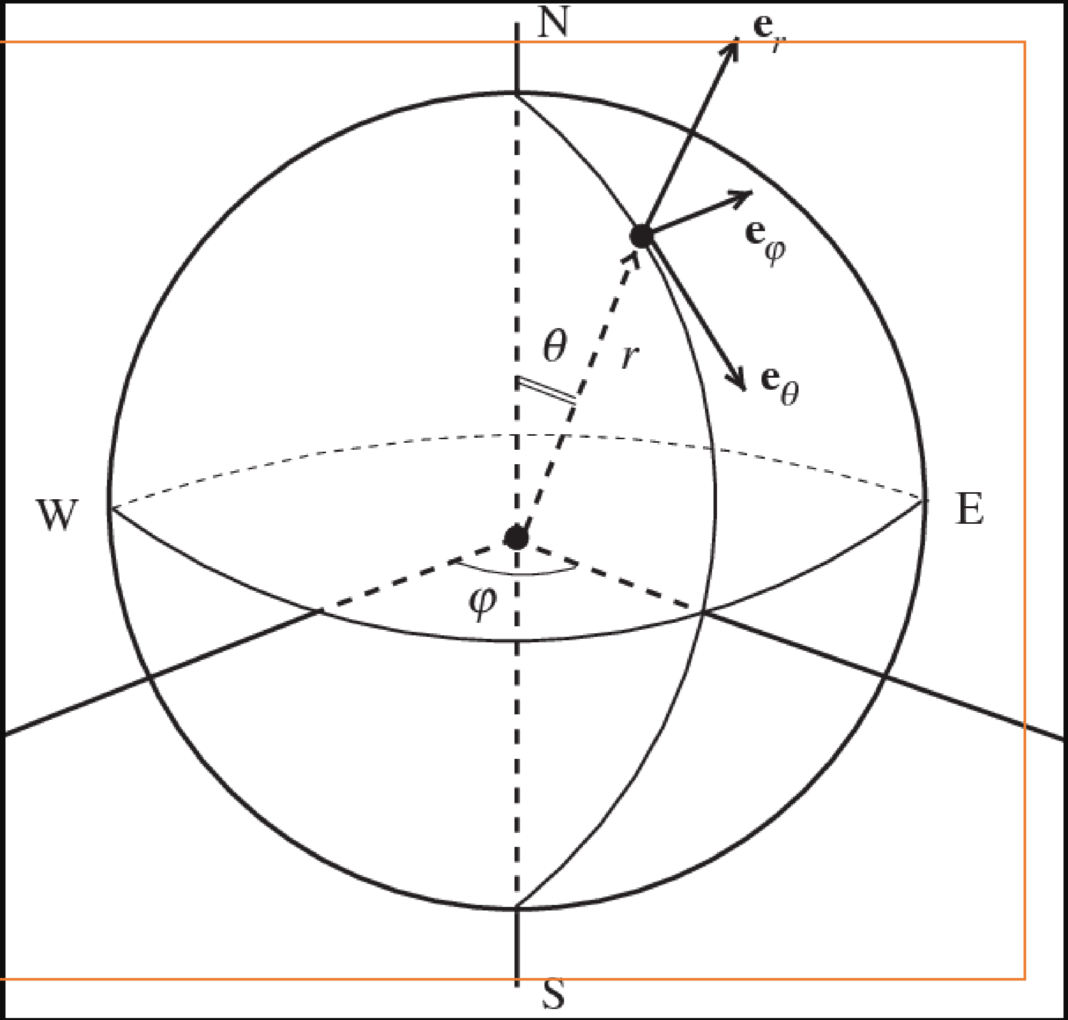
- The following are the different kinds of geometries possible:

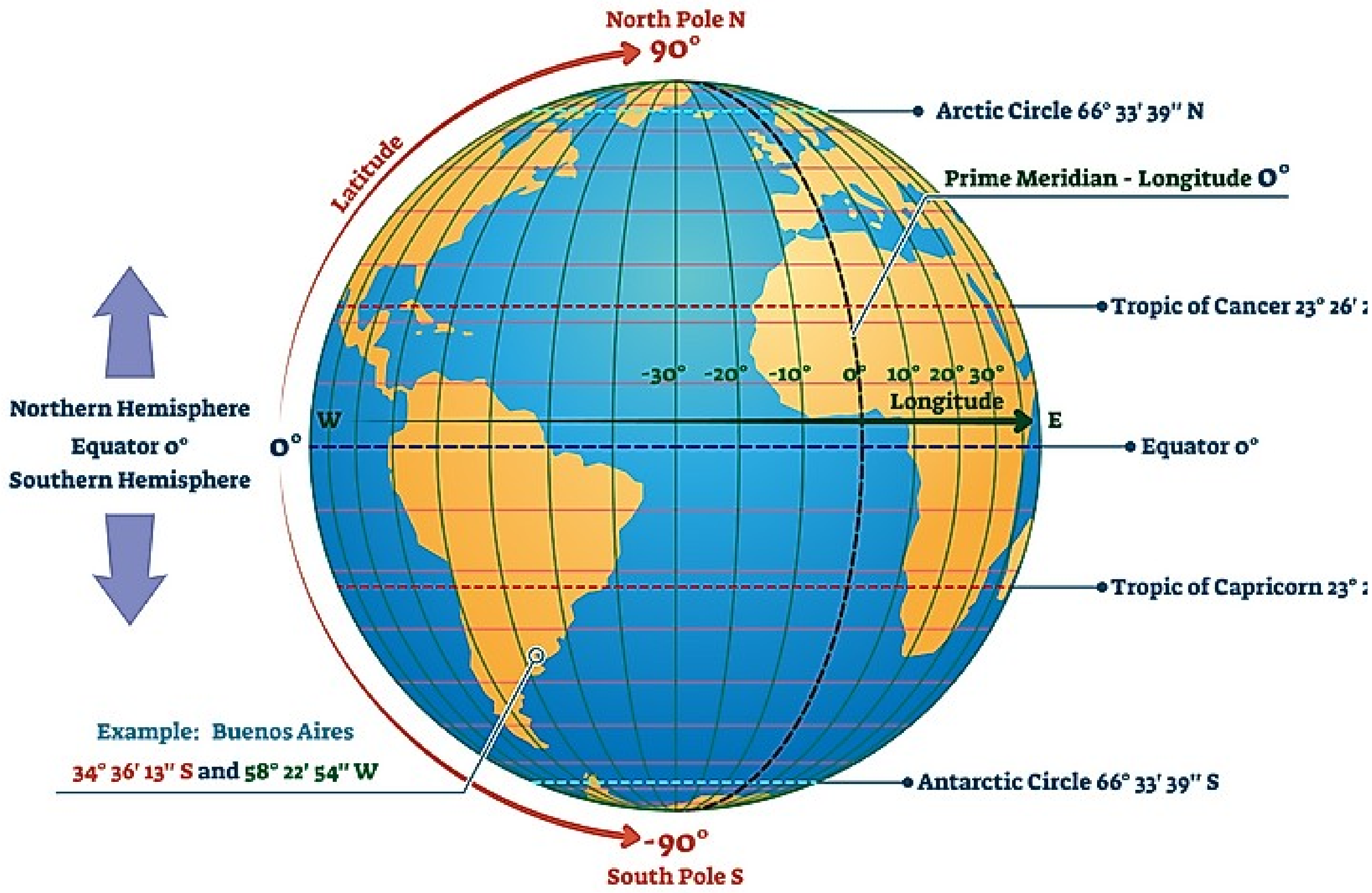


In this talk we will first focus on spherical geometry.

Coordinate systems in spherical geometry:

- The spherical coordinate system is the most widely used coordinate system for spherical geometry.
- These two coordinates are two angles, one is the polar angle and the other is the azimuthal angle.
- The polar angle runs from 0 to π and the azimuthal angle runs from 0 to 2π.





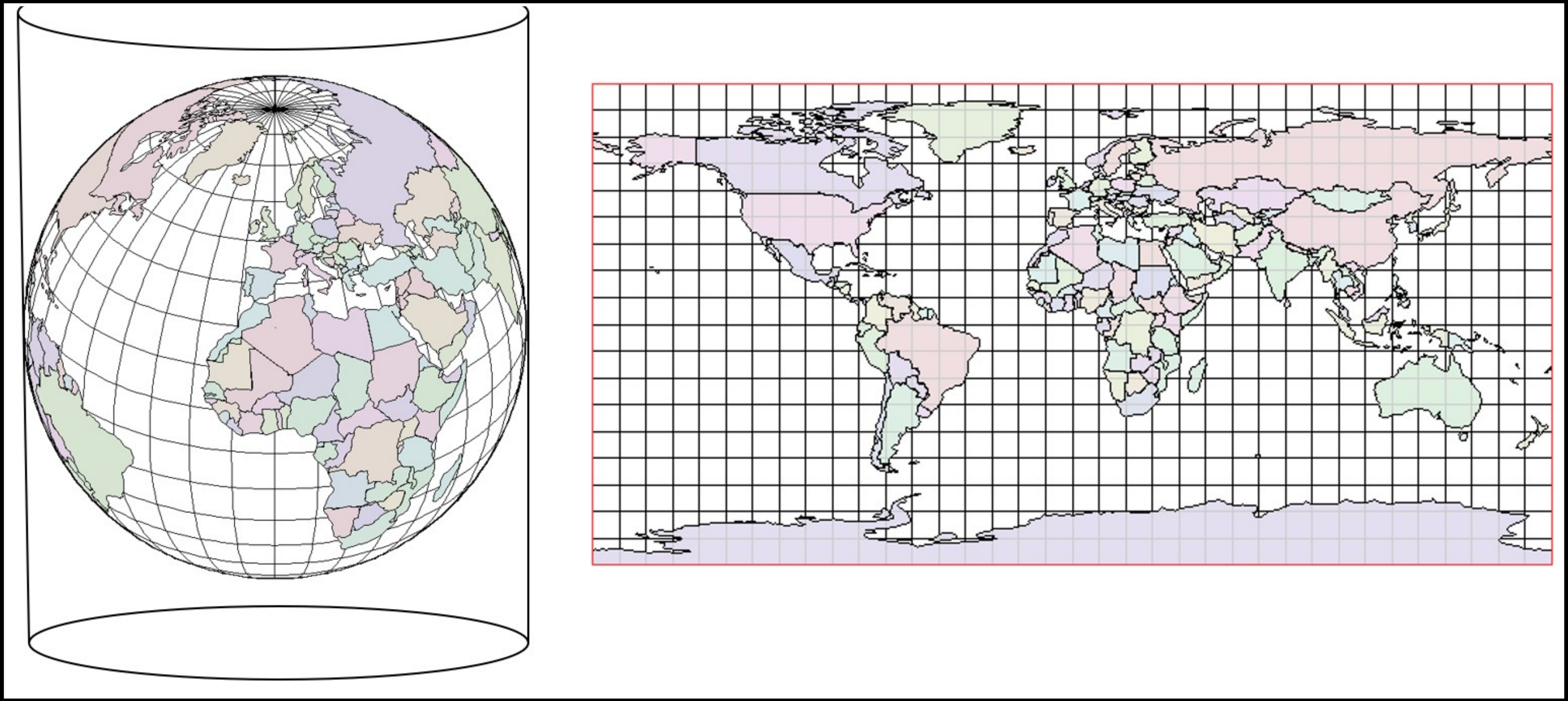
- A variant of the spherical coordinate systems is the latitude-longitude system used in geography.
- Here the angles are in degrees and the polar angle goes from -90° to 90°.

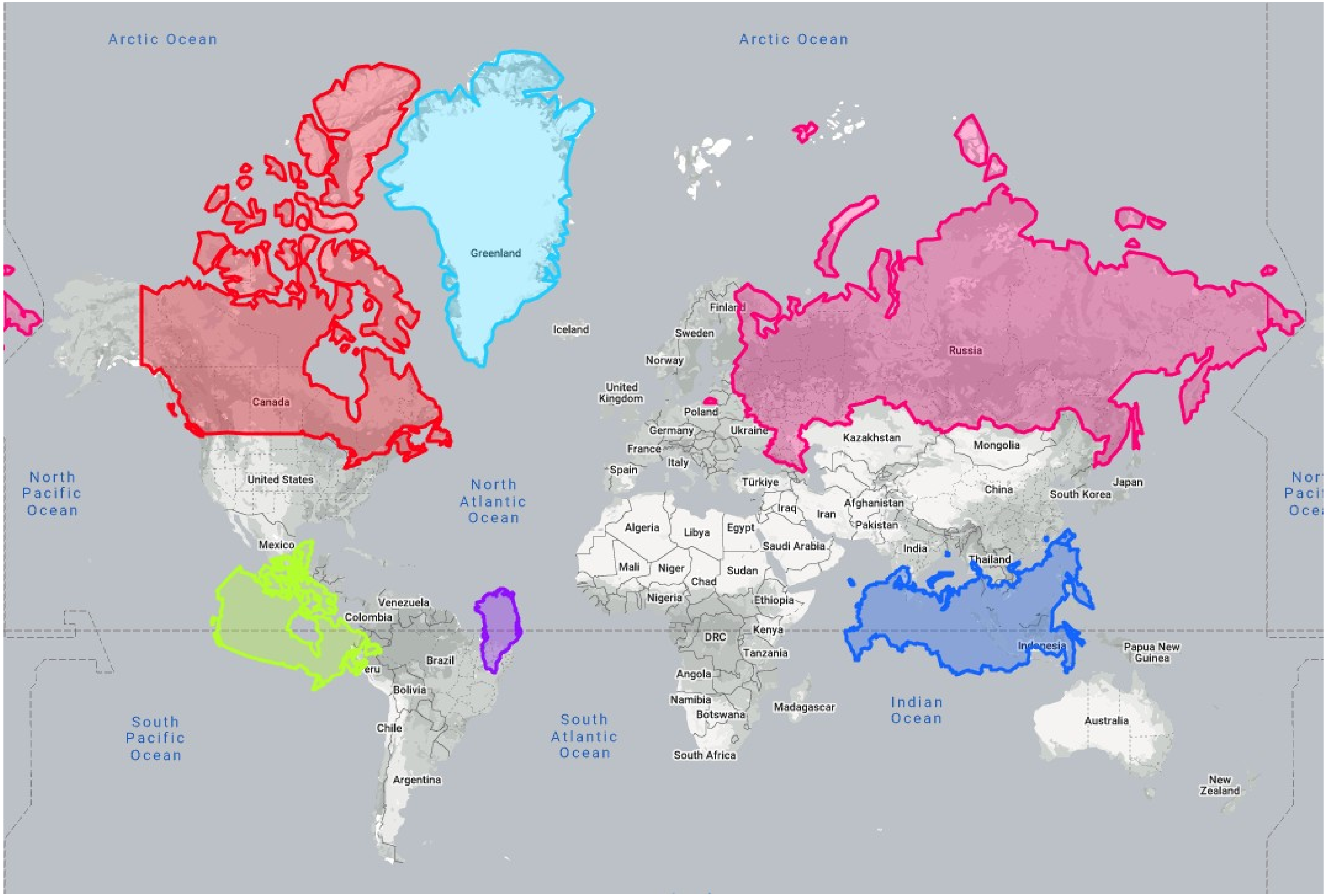
- The coordinates of our office are 18.5636558 and 73.7662159.
- This means that we are ~18.6° North of the Equator and ~73.8° East of London.

You can flatten out the spherical coordinate system into the flat plane, but no matter your method, you will always end with distortions in lengths, area or angles.

Below is the Mercator projection, here the latitude circles around the poles are stretched to become the same length as that of the latitude circle at the equator.

When we look at this flat map on the right, we cannot simply draw a straight line between two points on it as we learned to draw on the cartesian plane in school.



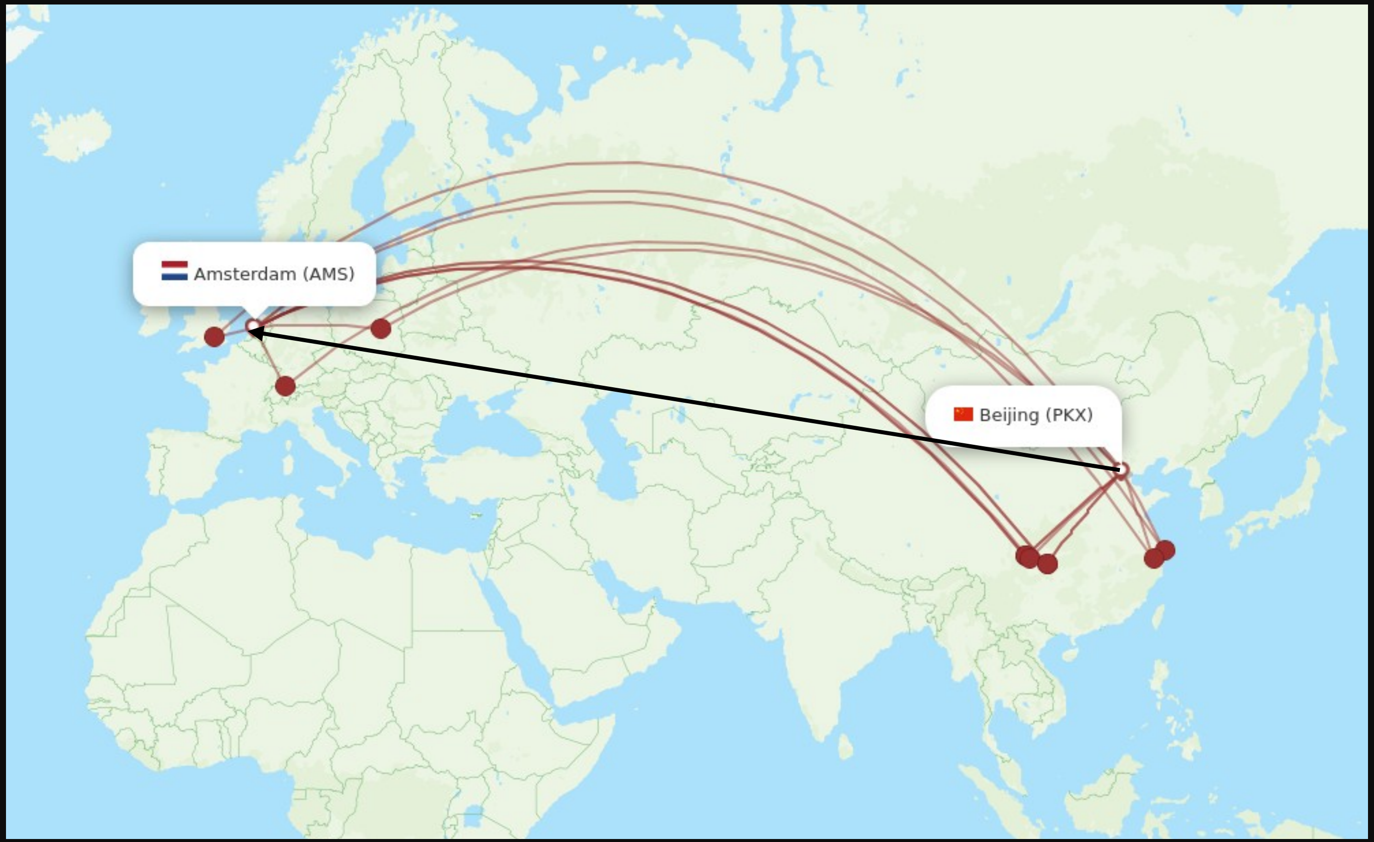


Here we see the extent of distortion caused by a Mercator projection.

The countries towards the North are more and more exaggerated in terms of lengths and area; here we see them as they would look like if they were near the equator.

Source: thetruesize.com

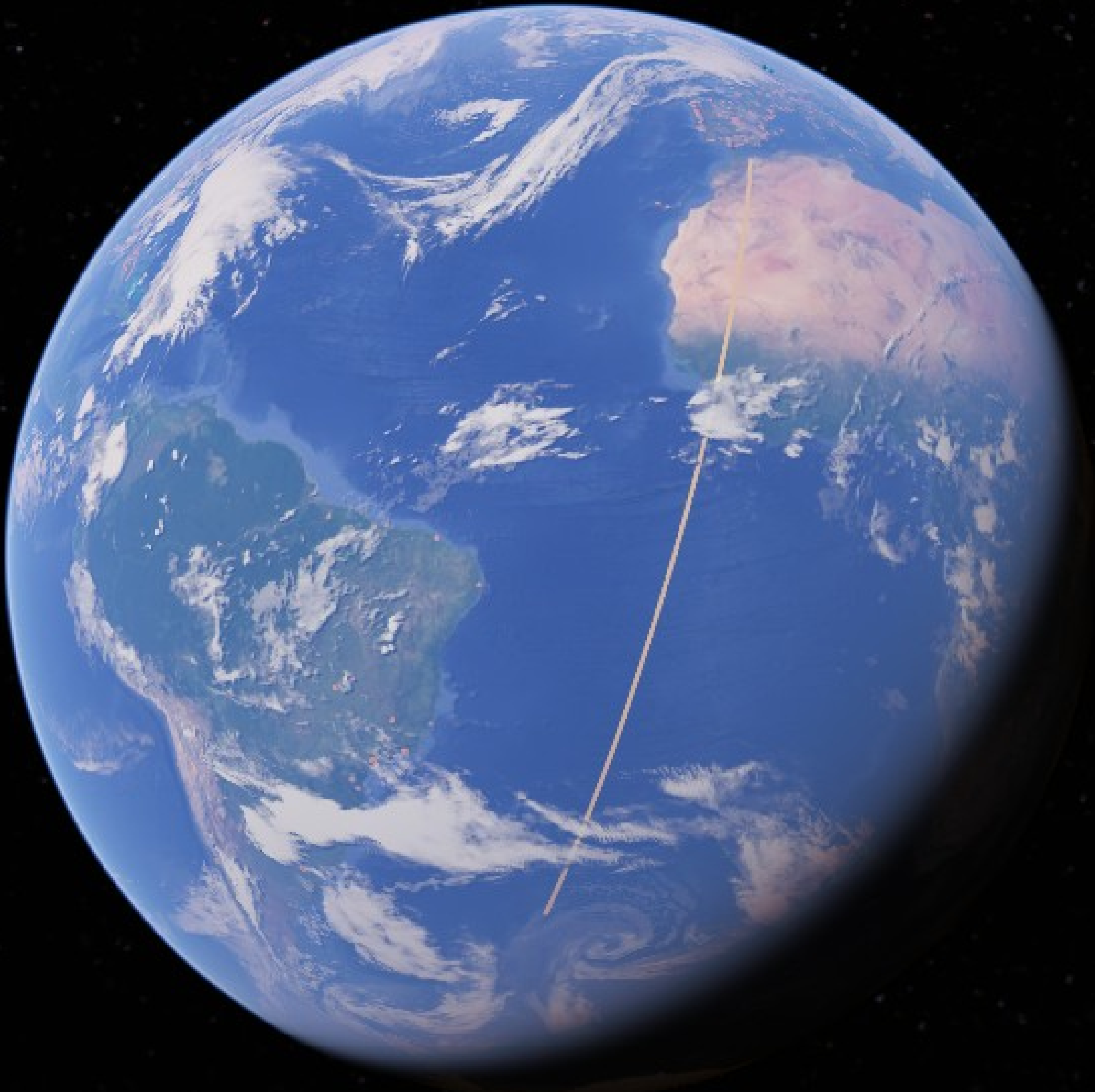
- These are the flight paths between Beijing and Amsterdam.
- The flights seem to be going on a curve.
- In this flat projection, the black line-segment seems to be the shortest distance as it is the straight line, but the red curve-segments are in fact the sh orter distances.



‘Straight lines’ on a spherical surface are segments along 'great'-circles.

A great-circle is a circle on the surface, whose center coincides with the center of the spherical surface.

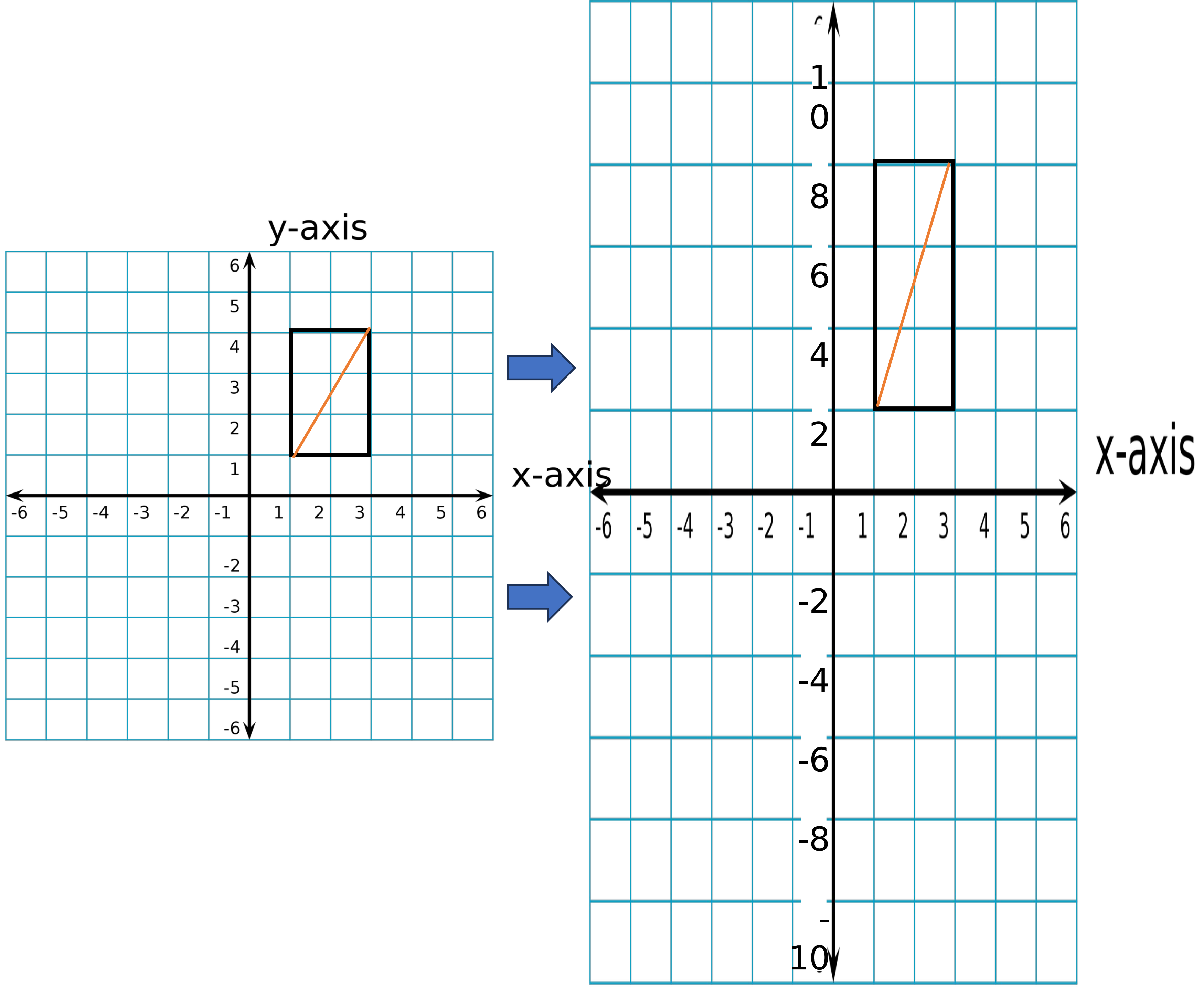
A path tracing the shortest distance is known as a geodesic (in any geometry) in general.



Now we must re-examine our Pythagorean formula for distance.

In this coordinate-system it does not seem to work.

But does it work in every coordinate system?

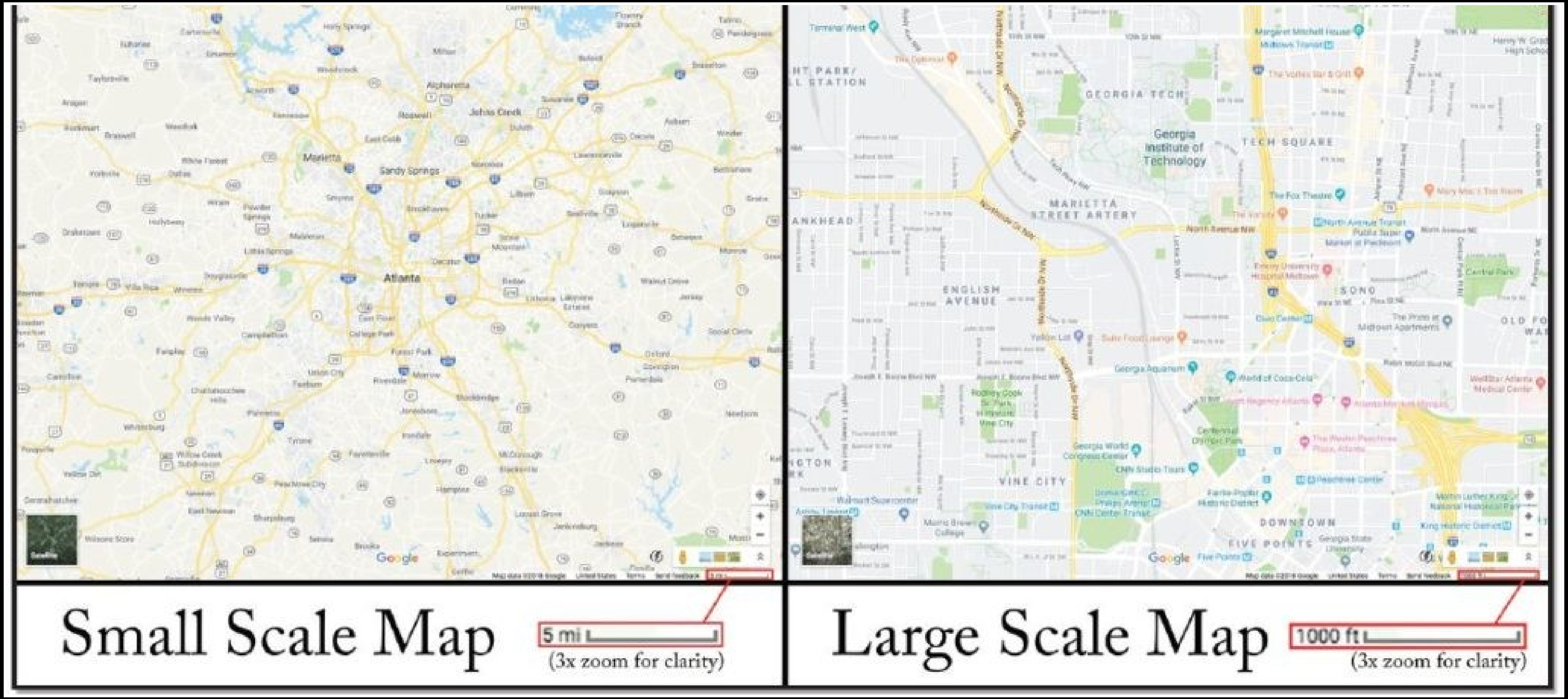


Now let us consider a coordinate system where 1 in the x-axis corresponds to 1 meter in real life and 1 in the y-axis corresponds to 2 meters in real life.

If there is a rectangle aligned with the grid in the parameter space with the **length a** along the x-axis and **length b** along the y-axis, what would the length of the diagonal in real life be?

It is apparent that the length of the diagonal will be

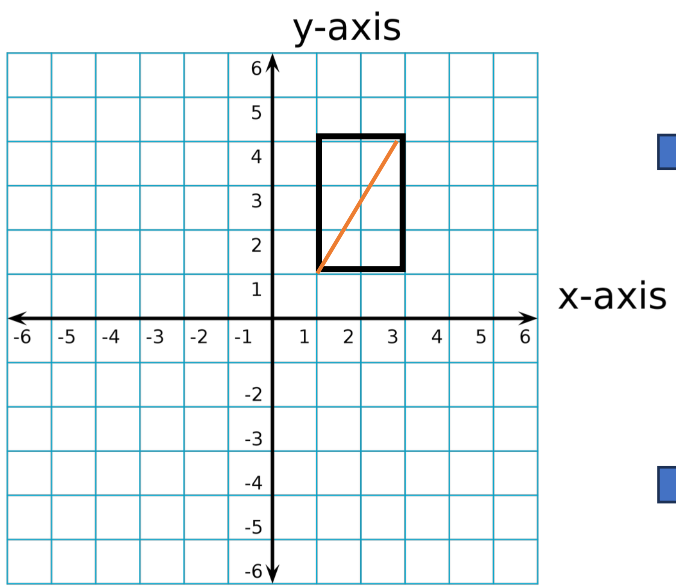
$$\sqrt{(a^2+(2b)^2)}$$

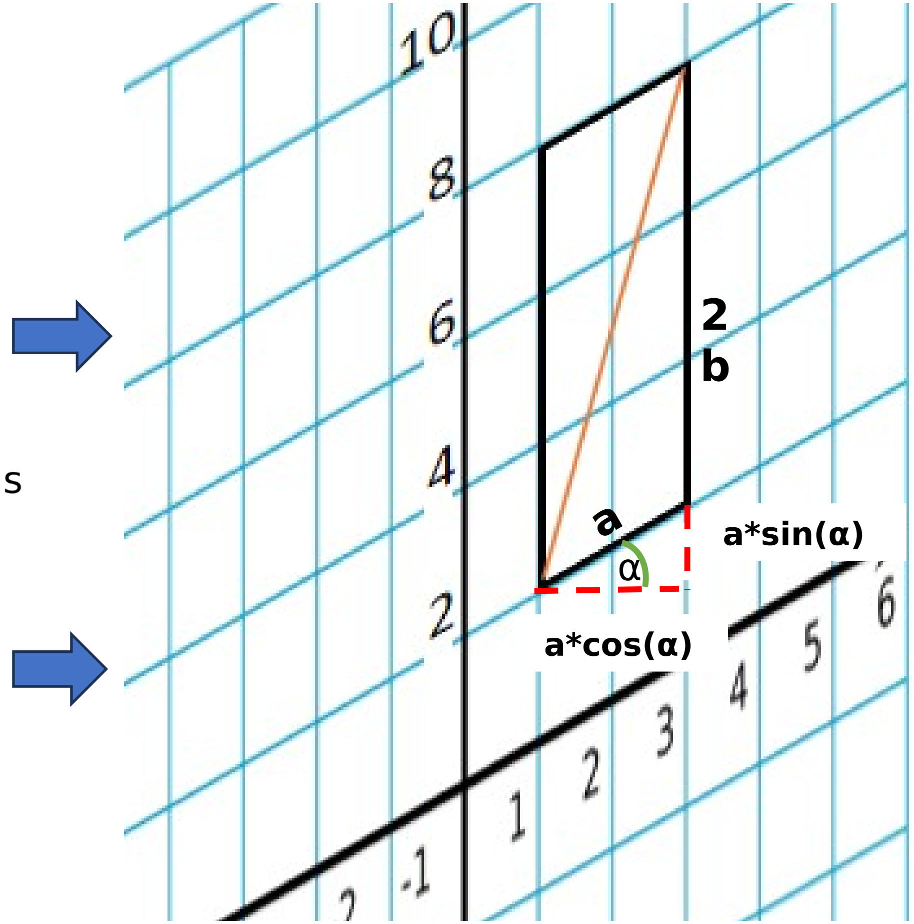


We are already used to converting from one set of coordinates to another set of coordinates by scaling them.

On a map, for example.

However, maps with a scale are generally of small areas where the Earth can be assumed to be flat and the distances cartesian.





Now in addition to being stretched in the y-direction, the grid has been skewed in the y direction by α, meaning the angle between the axes is now ß = π/2-α.

Same question:
If there is a rectangle aligned with the grid in the parameter space with the **length a** along the x-axis and **length b** along the y-axis, what would the length of the diagonal in real life be?

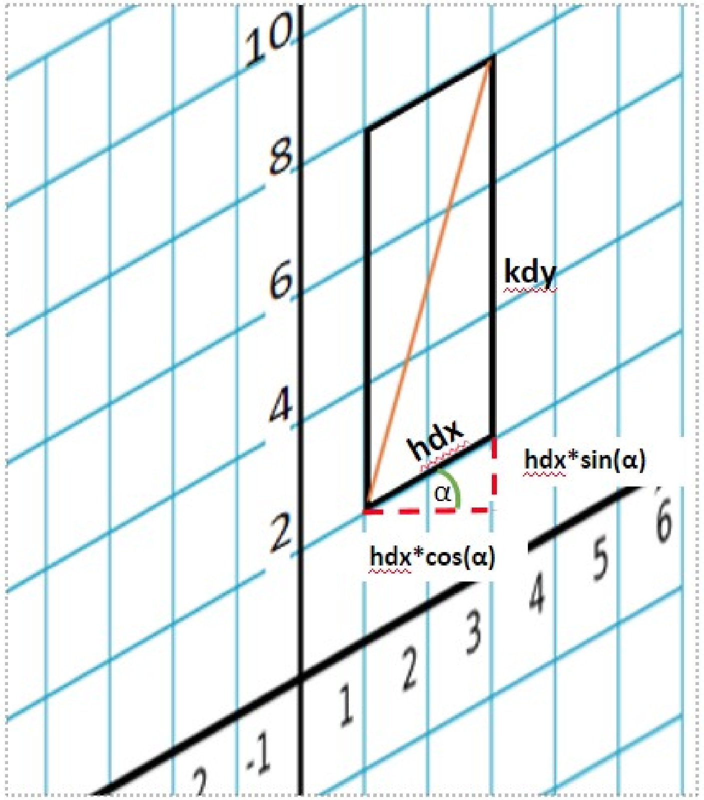
Using the cosine rule, we find the length to be:

$$\sqrt{((a\cos(\alpha))^2 + (2b + a\sin(\alpha))^2)}$$

Simplifying to:

$$\sqrt{(a^2 + (2b)^2 + 2a(2b)\cos(\beta))}$$

In general, if the x-axis is stretched by a factor of **h** and the y-axis is stretched by a factor of **k** and the angle between the axes is **α**; the square of the length of the diagonal of a rectangle with sides equaling dx along the x-axis and dy along the y axis will be:



$$ds^2 = h^2dx^2 + k^2dy^2 + 2(hdx)*(kdy)*cos(\alpha)$$

This can be simply written as this:

$$ds^2 = \begin{bmatrix} dx & dy \end{bmatrix} \begin{bmatrix} h^2 & hkcos(\alpha) \\ hkcos(\alpha) & k^2 \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$$

When h = k = 1 and α = 90°;

$$ds^2 = \begin{bmatrix} dx & dy \end{bmatrix} \begin{bmatrix} h^2 & hk\cos(\alpha) \\ hk\cos(\alpha) & k^2 \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$$

Reduces to:

$$ds^2 = \begin{bmatrix} dx & dy \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$$

Which is simply the Pythagorean formula we know and adore:

$$ds^2 = dx^2 + dy^2$$

- The matrix in the middle is known as the Metric Tensor. It tells you the infinitesimal length around a point in a certain coordinate system.
- It is equal to the Identity matrix in flat space.
- In general, the infinitesimal length between two points is given by:

$$ds^2 = \begin{bmatrix} dx & dy \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$$

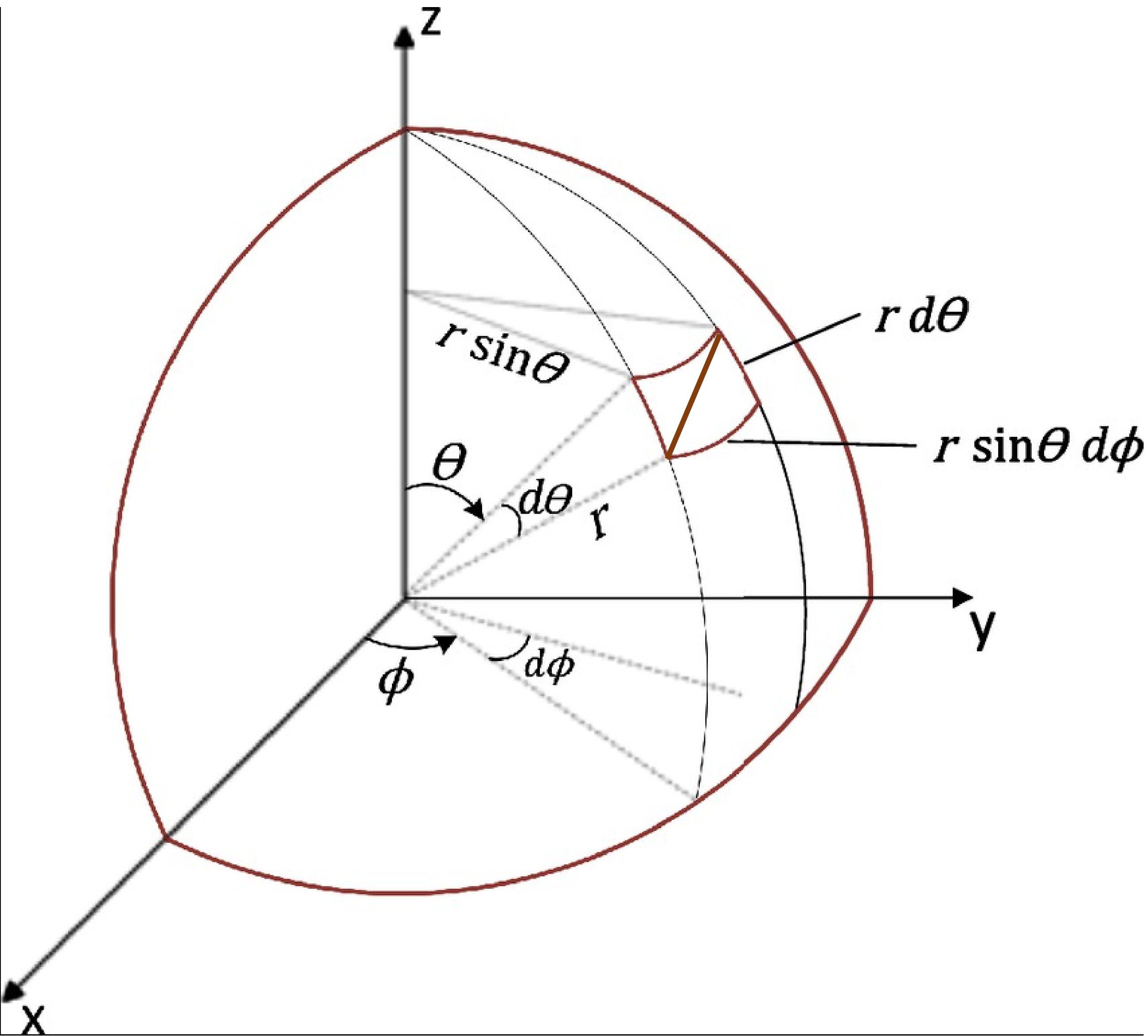
- Where g_11 and g_22 values represent the scale and the g_12 and g_21 values represent the angles between the axes.

- g_12 = g_21

- This metric tensor, represented by G_ij is not constant everywhere in the parameter space! We will soon see how.


That is okay!
But how does the metric tensor help us calculate the distance between two cities on Earth??
Toronto
Chicago
New York
Washington, D.C.
Atlanta
Houston
Havana
Santo Domingo
Dublin
London
Amsterdam
Brussels
Paris
Madrid
Lisbon
Algiers
Casablanca
Dakar

First, we find the diagonal line-element on the surface of a sphere using the spherical coordinates system.



We assume that the sphere appears flat when we are very close to it.

Now we want to calculate the length of the line element around a particular point (Θ,Φ).

From the construction the line element comes out to be:

$$ds^2 = r^2 d\theta^2 + r^2 sin^2\theta d\phi^2$$

In another words,

$$ds^2 = \begin{bmatrix} d\theta & d\phi \end{bmatrix} \begin{bmatrix} r^2 & 0 \\ 0 & r^2 sin^2\theta \end{bmatrix} \begin{bmatrix} d\theta \\ d\phi \end{bmatrix}$$

Notice that the anti-diagonal terms remain zero because the angle between the 'axes' remains constant everywhere at 90°.

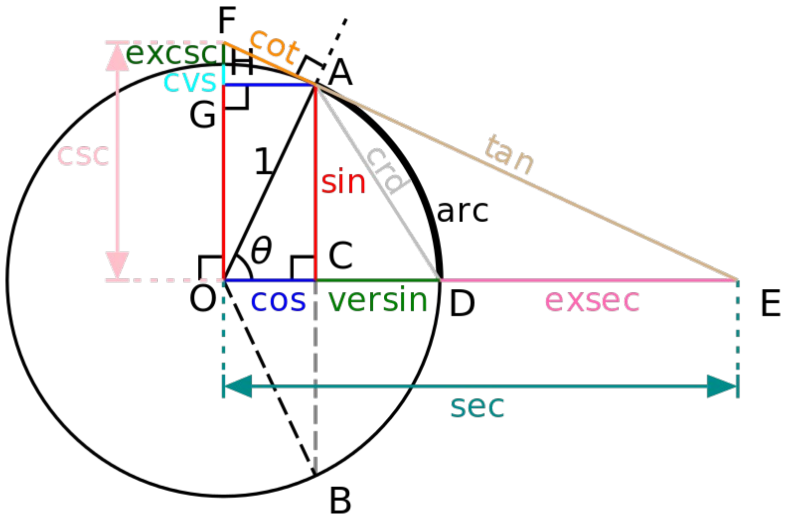
Distances are scaled with a constant along the polar angle and scaled with a sine along the azimuthal angle.

Calculating shortest distance between two points on a sphere

If we have two points: (θ1,φ1) and (θ2,φ2); how do we calculate the distance between them?

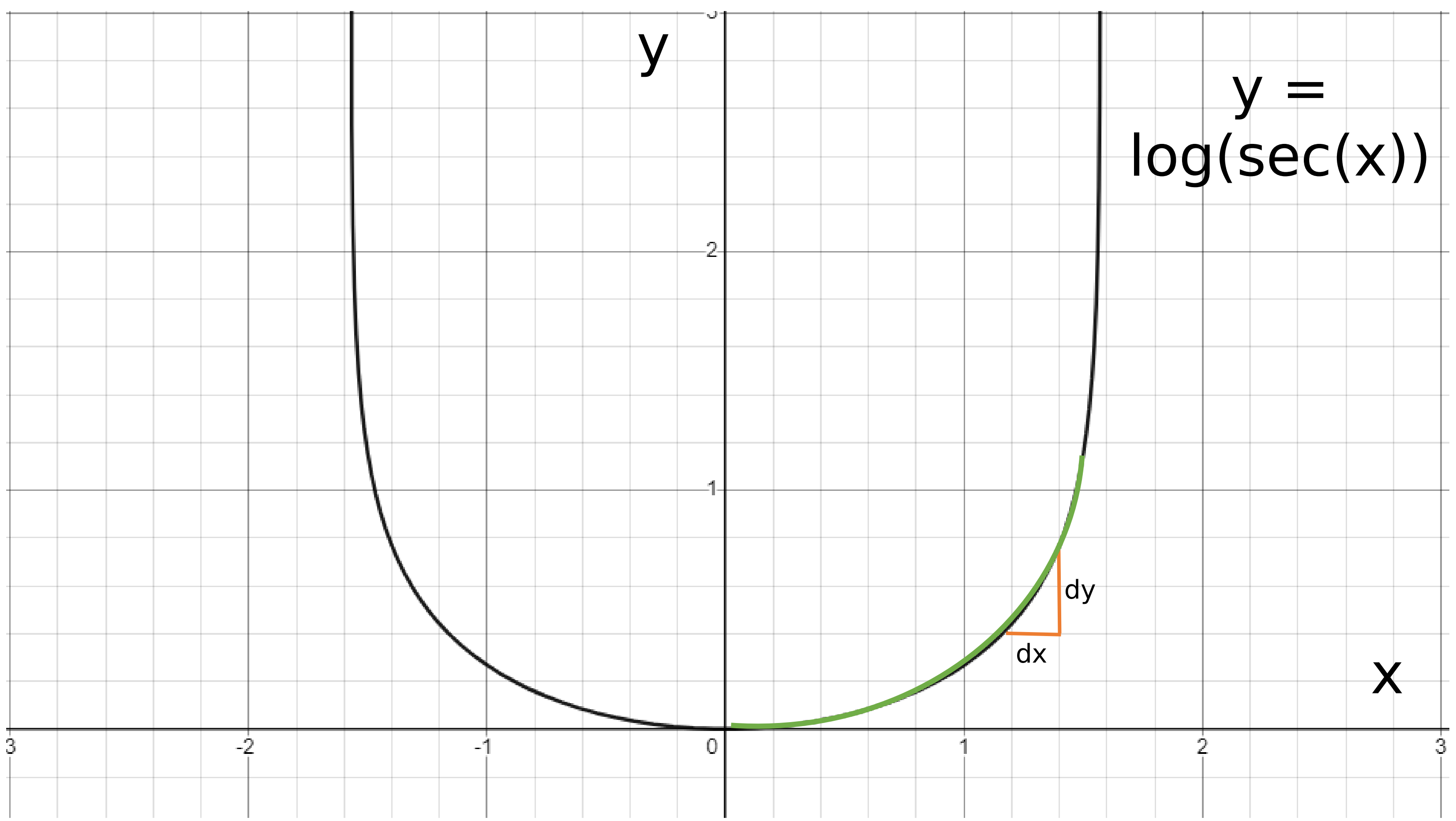
You might say that we can simply convert them to cartesian coordinates, find the length of the chord connecting them and then find the angle subtending it and multiply that angle by r.

We however want to calculate only using the two-dimensions and the metric tensor, you will see why later in this presentation.



Let us practice measuring distance along a curve in flat space first.

Let us calculate the distance along this curve from x = 0 to x = 1.5, the function is chosen for convenience.



The infinitesimal line element is:

$$ds = \sqrt{dx^2 + dy^2}$$

So, the length along the curve

$$s = \int_{x=0}^{1.5} \sqrt{dx^2 + dy^2}$$

$$s = \int_{x=0}^{1.5} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

Here, dy/dx = tan(x) ∴

$$s = \int_{x=0}^{1.5} \sqrt{1 + tan^2(x)}\ dx$$

$$s = \int_{x=0}^{1.5} sec(x)dx$$

$$s = ln|sec(x) + tan(x)|\Big|_0^{1.5}$$

$$s = 3.3406$$

Similarly, to find the (minimum) distance between two points on the surface of the Earth we have to integrate the line-element along the geodesic.

Remember: $ds^2 = r^2 d\theta^2 + r^2 sin^2\theta d\phi^2$

$$s = \int \sqrt{r^2 d\theta^2 + r^2 sin^2\theta d\phi^2}$$

Integrating over the azimuthal angle as it covers the entire circle:

$$s = r^2 \int_{\phi_1}^{\phi_2} \sqrt{\left(\frac{d\theta}{d\phi}\right)^2 + sin^2\theta} \; d\phi$$

Please note that this integral can be used for any path between these two points. As we need the shortest path, we are going to choose the equation of the great circle:

$$cot\theta = -Acos(\phi - \phi_0)$$

Where A is cotangent of the minimum polar angle and ϕ_0 is the azimuth where the circle intersects the equator.

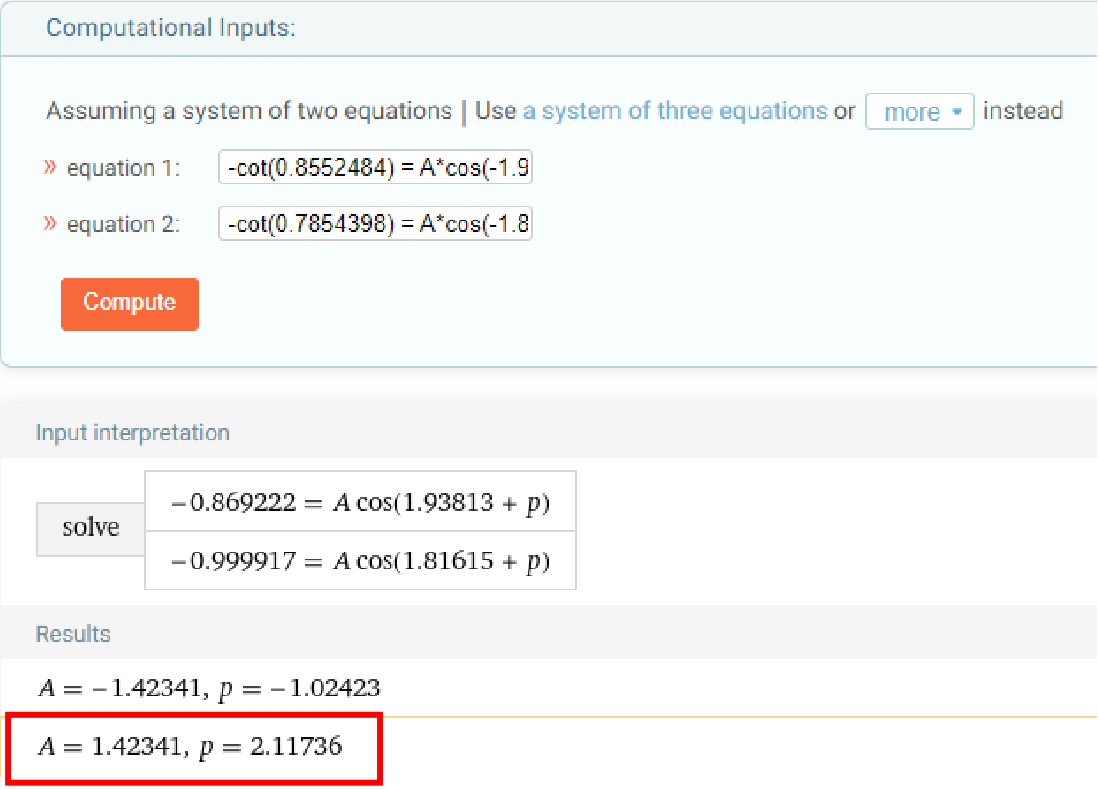
Now we need to find the equation of the great circle between the diagonal ends of Wyoming!

After putting in the derivative, of the integral simplifies to:

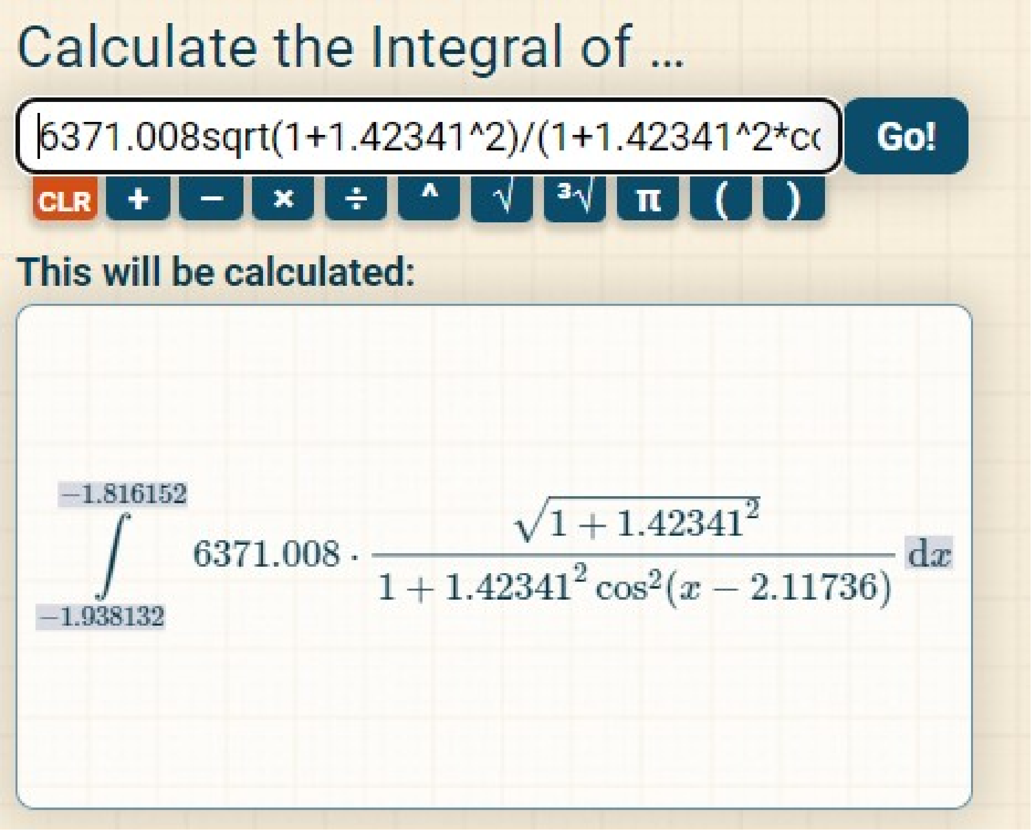
$$s = r\int_{\phi_1}^{\phi_2} \frac{\sqrt{A^2+1}}{A^2 cos^2(\phi-\phi_0)+1} d\phi$$

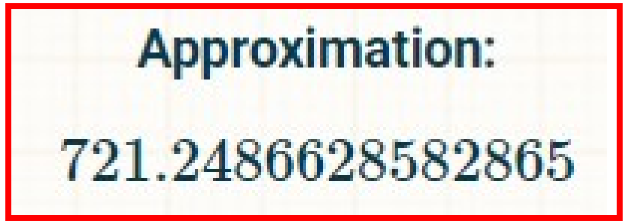
Now let us compute the length of Wyoming's diagonal. First, we need to find the appropriate A and ϕ_0 for the one and only great circle that passes through these two points.

I used Wolfram Alpha to solve the simultaneous trigonometric equations.

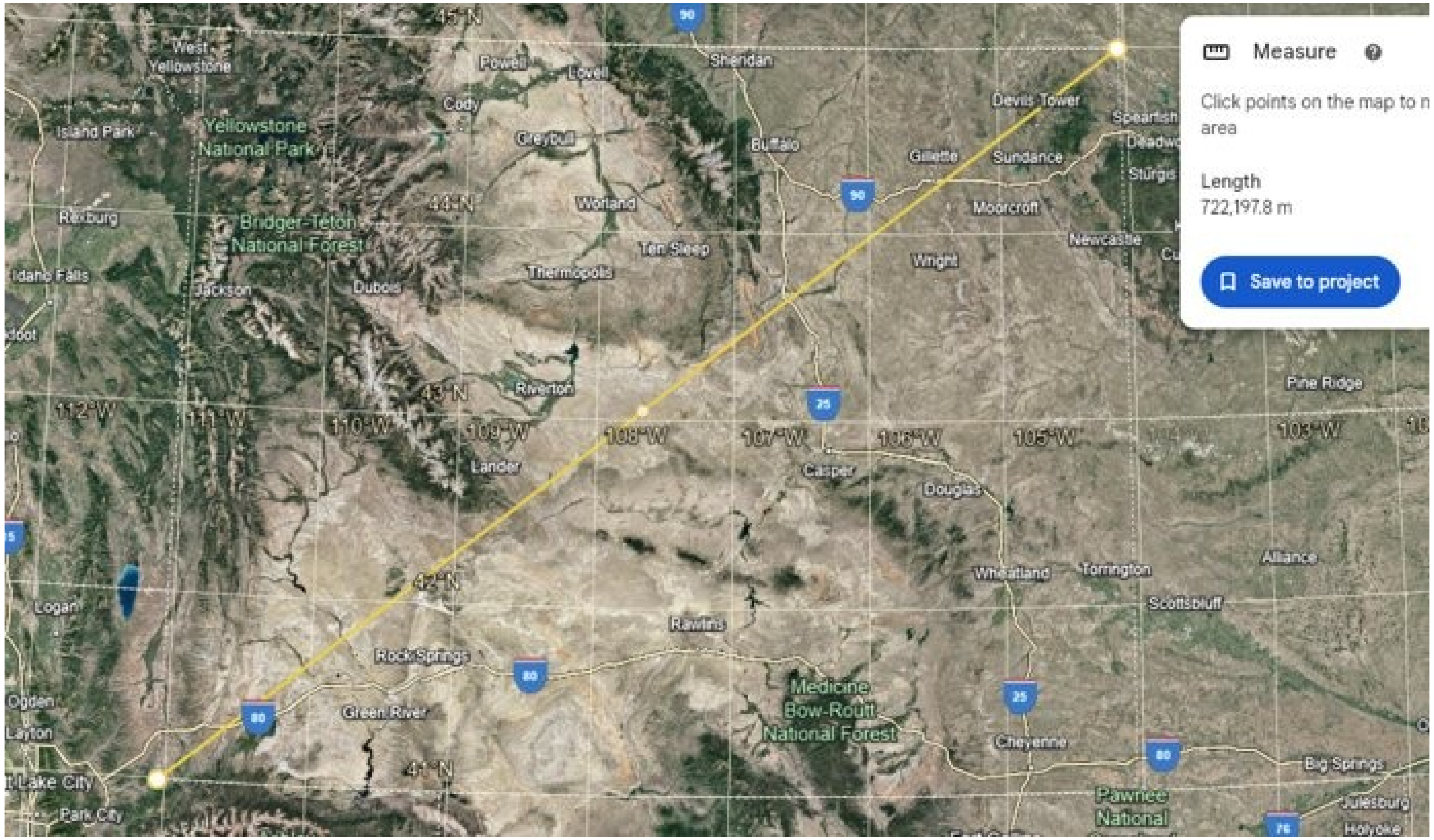


Then I used integral-calculator.com to evaluate the definite integral:





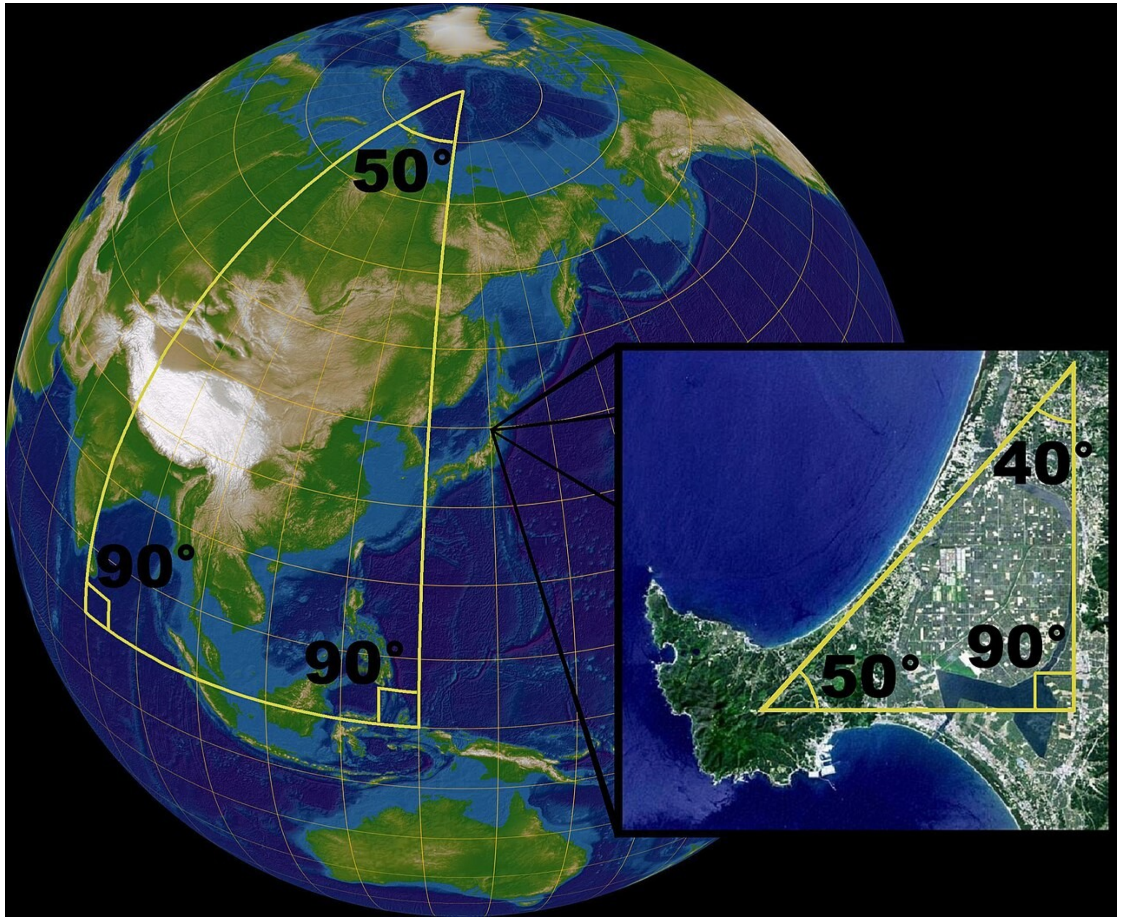
721.2 km is just 1km short of the actual length of 722.2. The error may come from the fact that the Earth is not a perfect sphere, as it is flatter at the top and thicker at the Equator.



And the sum of internal angles of a triangle on a spherical surface comes out to be:

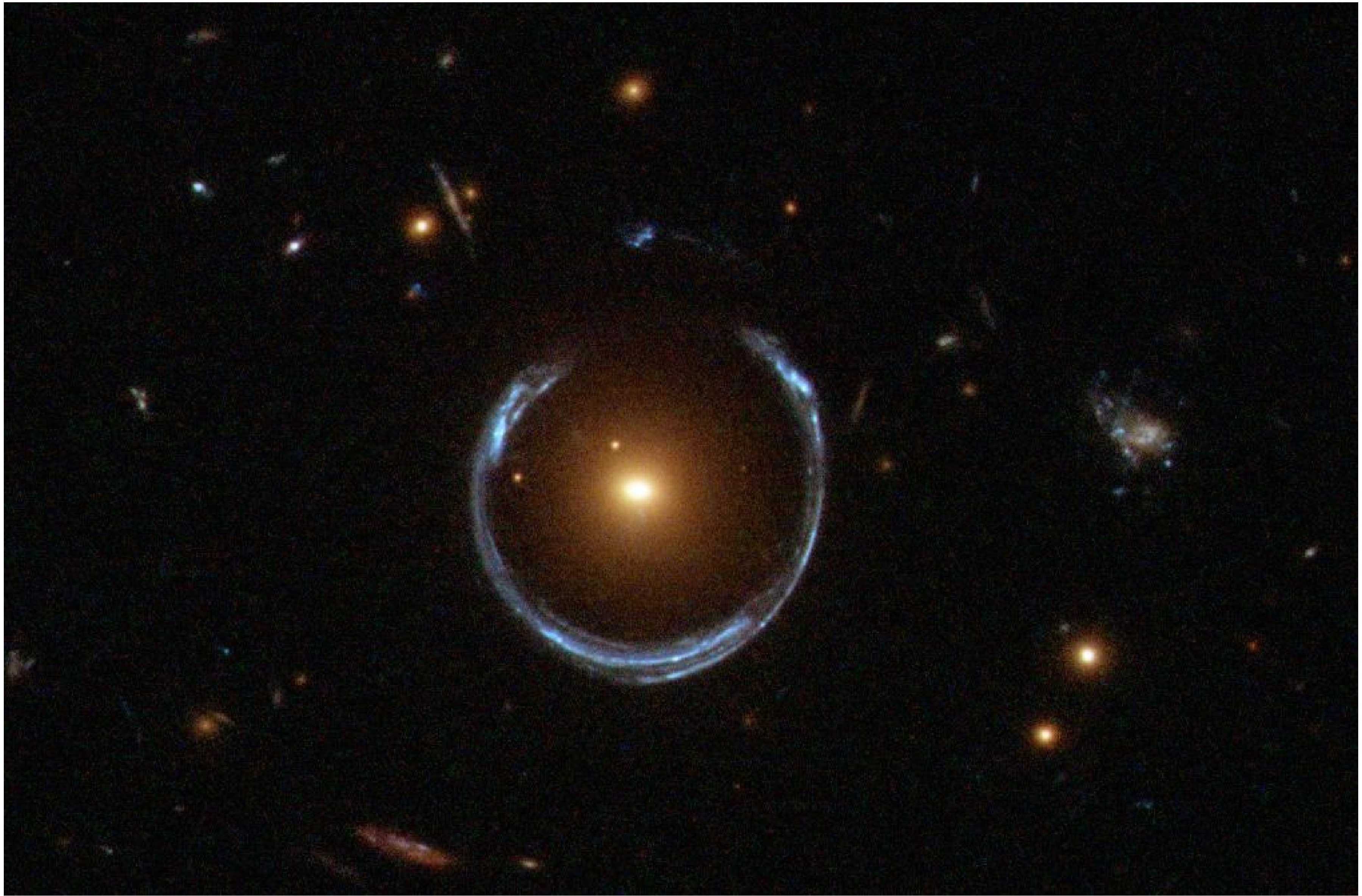
$$\sum_{vertex1}^{vertex3} angle = \pi + \frac{A}{r^2}$$

Where A is the area of the triangle and r is the radius of the sphere.
Notice that it is equal to π when r→∞.

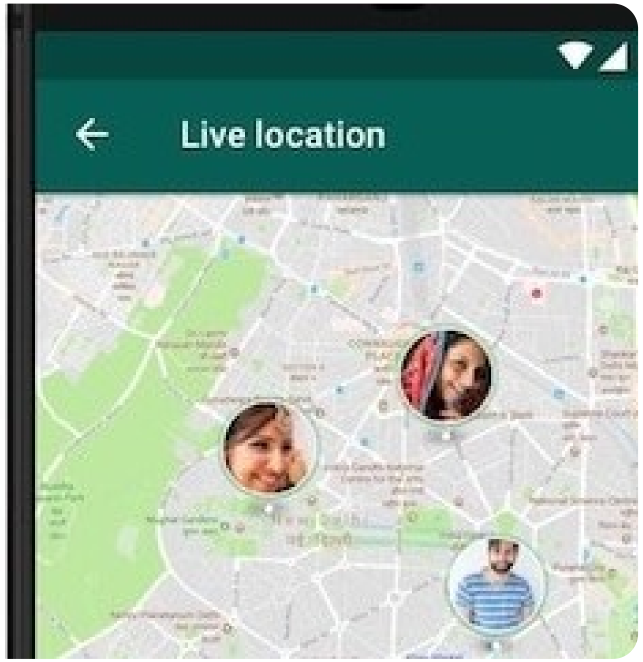


But is that all?
Are there any other real-life uses of non-Euclidean Geometry?

This is a real image taken by the Hubble Space Telescope, the blue arc you see is actually a flat galaxy behind the yellow galaxy. Can Euclidean geometry explain this?



But forget objects in deep space!



You wouldn't have been able to watch TV as a kid without non-Euclidean geometry!

You wouldn't be able to share your live location with a friend today if not for non-Euclidean geometry!

But what does non-Euclidean geometry have to do with TVs or Geometry?

Let us expand our minds a little.

Let us say that you want to meet up with your friend.

What kind of things do you need to discuss so that you two end up meeting?

Very simply, you need to give them a time and a place.

So, in terms of C++ programming, you have to provide them with 4 double values: (x,y,z,t).

(Actually, on the Earth you'd need to give latitude, longitude and the altitude but you get my point)

With three spatial dimensions you can describe a **location**, but when you add time as the fourth dimension, you can describe an **event**.

An event could be something like a firecracker bursting at a certain location.

So now we have a 4D space, the next logical thing to do is to define the length between two events in this 4D space.

We can simply define it to $ds^2 = dx^2 + dy^2 + dz^2 + dt^2$

The first issue we have here is that distance is measured in meters and time is measured in seconds, so we cannot simply add their squares.

So how do we convert time into space? We need to multiply with some sort of a inter-dimensional scaling factor, and it needs to have the dimensions of distance/time.

Distance/Time has the dimensions of speed.

Obviously, we can pick any speed we want, but we want it to be meaningful and independent of the observer and their frame of reference.

It turns out, such a speed exists, and it is known as "the speed of light in vacuum" and represented by just 'c'.

It was discovered that no matter our own velocity, we always measured the speed of light in vacuum to be the same: 299,792,458 m/s (exactly).

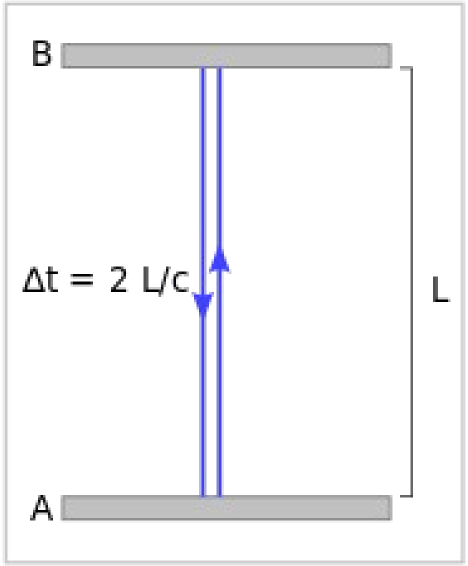
We can never know our own speed absolutely; we can only define it relative to other objects.

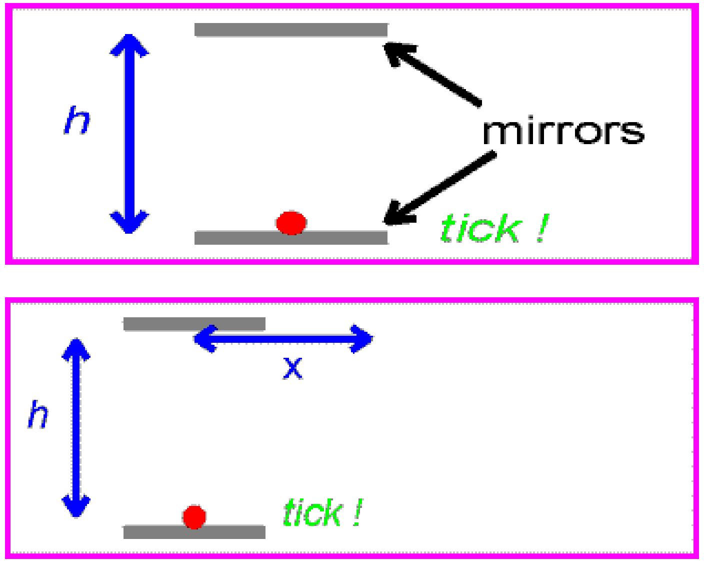
But we can always measure the speed of light and find the same value every time.

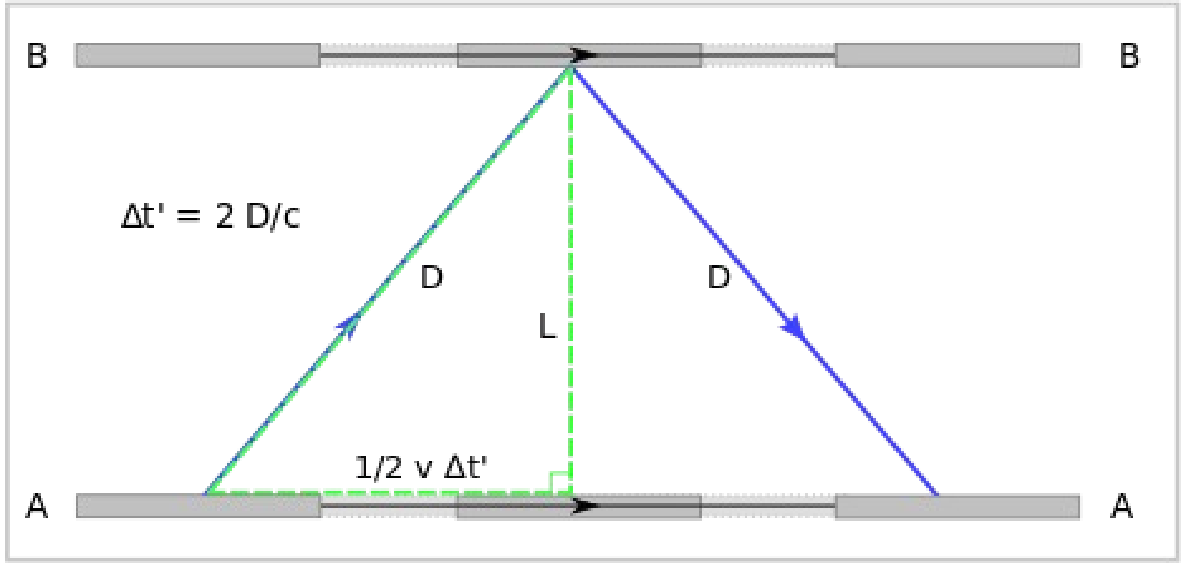
This is the basis for Einstein's theory of relativity.

What are the consequences of this postulate that the speed of light in vacuum remains constant in all the frames?

Let us think about a simple thought experiment:



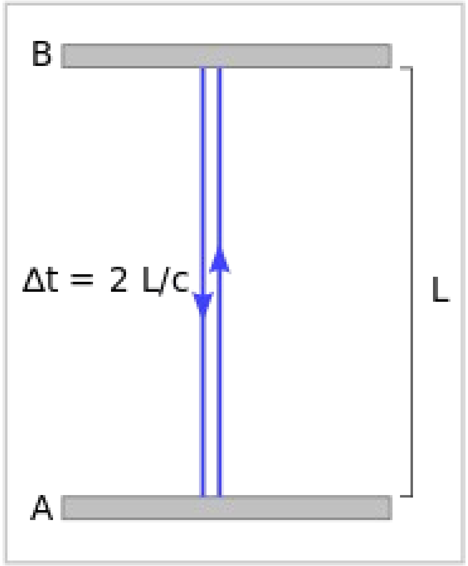




Here an observer standing inside a moving train car sends a ray of light towards the roof of the car where a mirror is placed which reflects the light ray back.
The observer then calculates the speed of light as 2L/t.

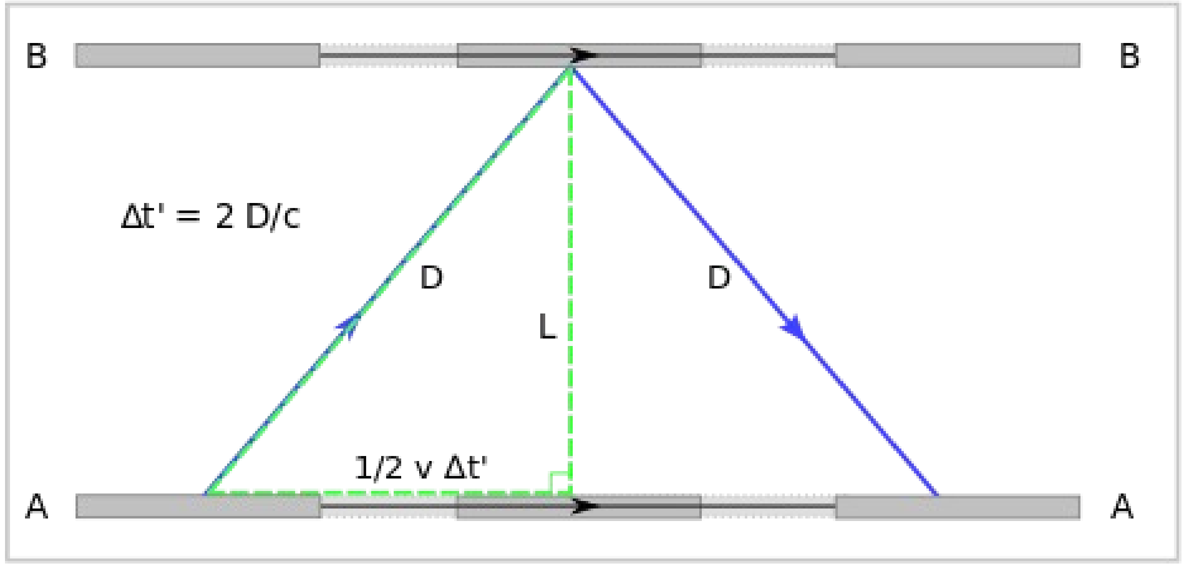
An observer standing on the platform sees the light ray travel in the manner shown as above. He calculates the speed of light as 2D/t. But clearly D is larger than L so the observers will not agree on the speed of light.

This is only possible if time flows slowly inside the moving train as compared to the platform.



So now let us calculate how many seconds in one frame equal to 1 second in the other frame.

We will work with assumption that both the observers will calculate the same speed of light in both the frames.



$$2L = ct$$

$$2\sqrt{L^2 + \frac{v^2 t'^2}{4}} = ct'$$

$$4L^2 + v^2 t'^2 = c^2 t'^2$$

$$c^2 t^2 + v^2 t'^2 = c^2 t'^2$$

$$t'^2 \left(c^2 - v^2\right) = c^2 t^2$$

$$t' = \frac{t}{\sqrt{1 - \frac{v^2}{c^2}}}$$

This is time dilation!

Let's look at the following equation from the previous slide:

$$c^2t^2 + v^2t'^2 = c^2t'^2$$

Let's divide both sides by 4 and rearrange:

$$c^2\frac{t^2}{4} = c^2\frac{t'^2}{4} - v^2\frac{t'^2}{4}$$

Let's add -L² to both sides and rearrange:

$$c^2\frac{t^2}{4} - L^2 = c^2\frac{t'^2}{4} - \left(L^2 + v^2\frac{t'^2}{4}\right)$$

$$c^2\frac{t^2}{4} - L^2 = c^2\frac{t'^2}{4} - \left(L^2 + v^2\frac{t'^2}{4}\right)$$

It looks like a formula for a spacetime distance that is getting preserved across transformation within frames!

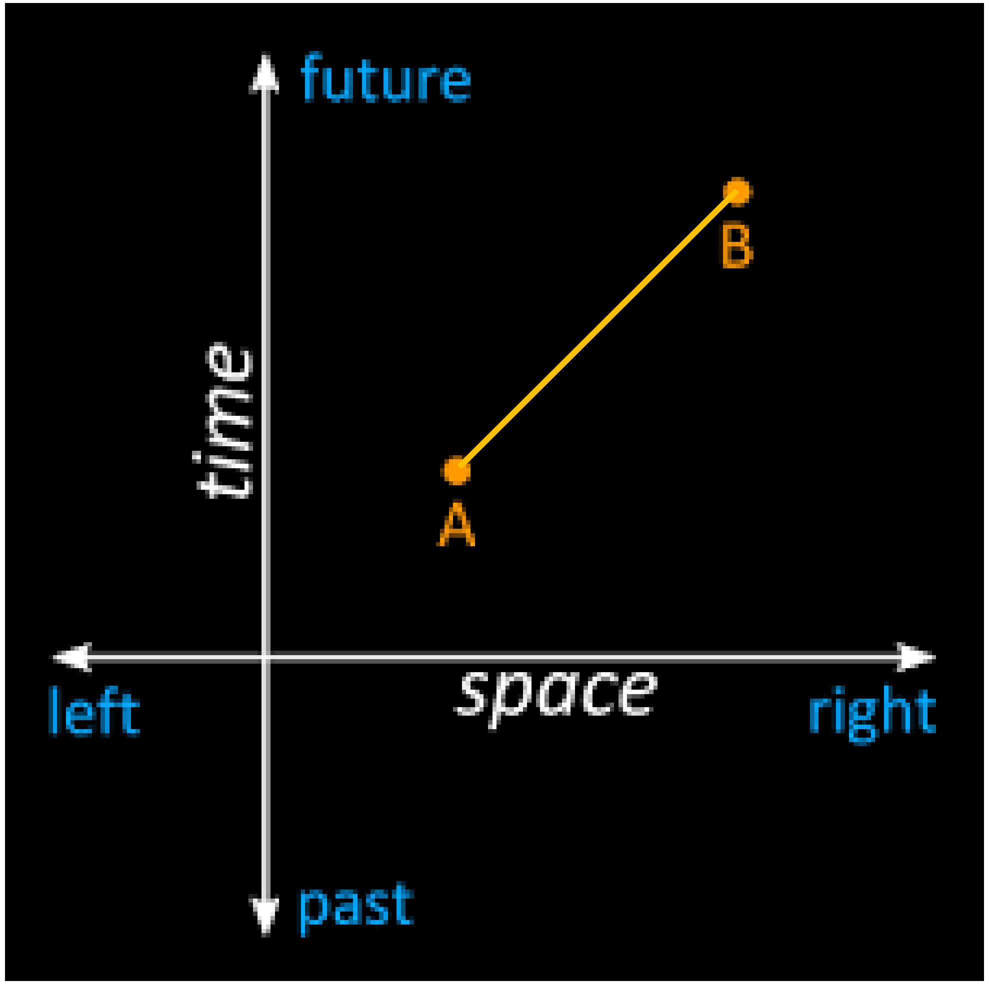
So, the distance in the spacetime is defined as:

$$\Delta s^2 = c^2 \Delta t^2 - \Delta x^2$$

If we tried to make it "Euclidean" by defining it as:

$$\Delta s^2 = c^2 \Delta t^2 + \Delta x^2$$

We would have time contraction instead of time dilation and that would violate the postulate of the special relativity that the speed of light in vacuum needs to be the same in every frame of reference



Hence the distance or the line element in the 4D spacetime is defined as the following:

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

In other words:

$$ds^2 = [dx\ dy\ dz\ dt]\begin{bmatrix} c^2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} dx \\ dy \\ dz \\ dt \end{bmatrix}$$

Light travels along the path along which *this* distance is the shortest.
This distance between two events is the same in every inertial frame.

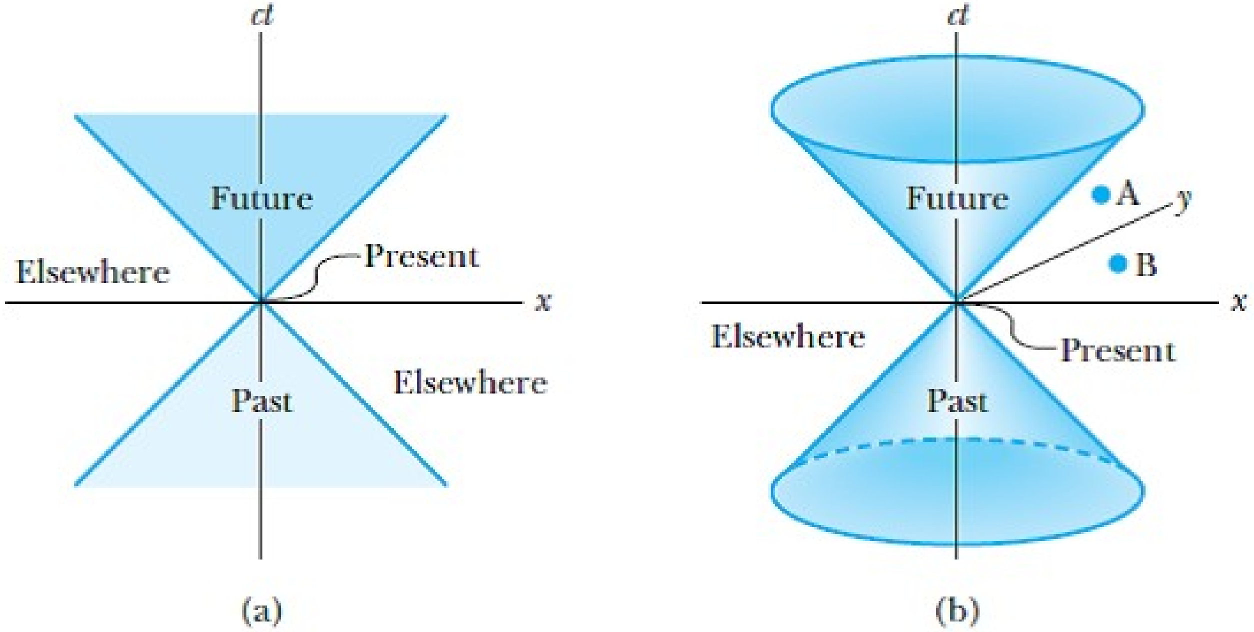
Please note that some physicists use a positive sign for the spatial components and a negative sign for the time component.

If a particle has no mass, it travels at the speed of light.
Since every other particle travels slower; for a given time t, the distance travelled x(=vt) by that particle will always be less than ct.

Hence if the spacetime distance between two events is positive, it is possible for a particle to travel from one event to another.

If it is zero, only light can connect these two events.

If it is negative, it is not possible that those two events have any connection whatsoever.



Although the surface of the Earth is curved and hence the shortest distance along it is curved, it is technically possible to dig through the Earth and take a shorter path to the other point.

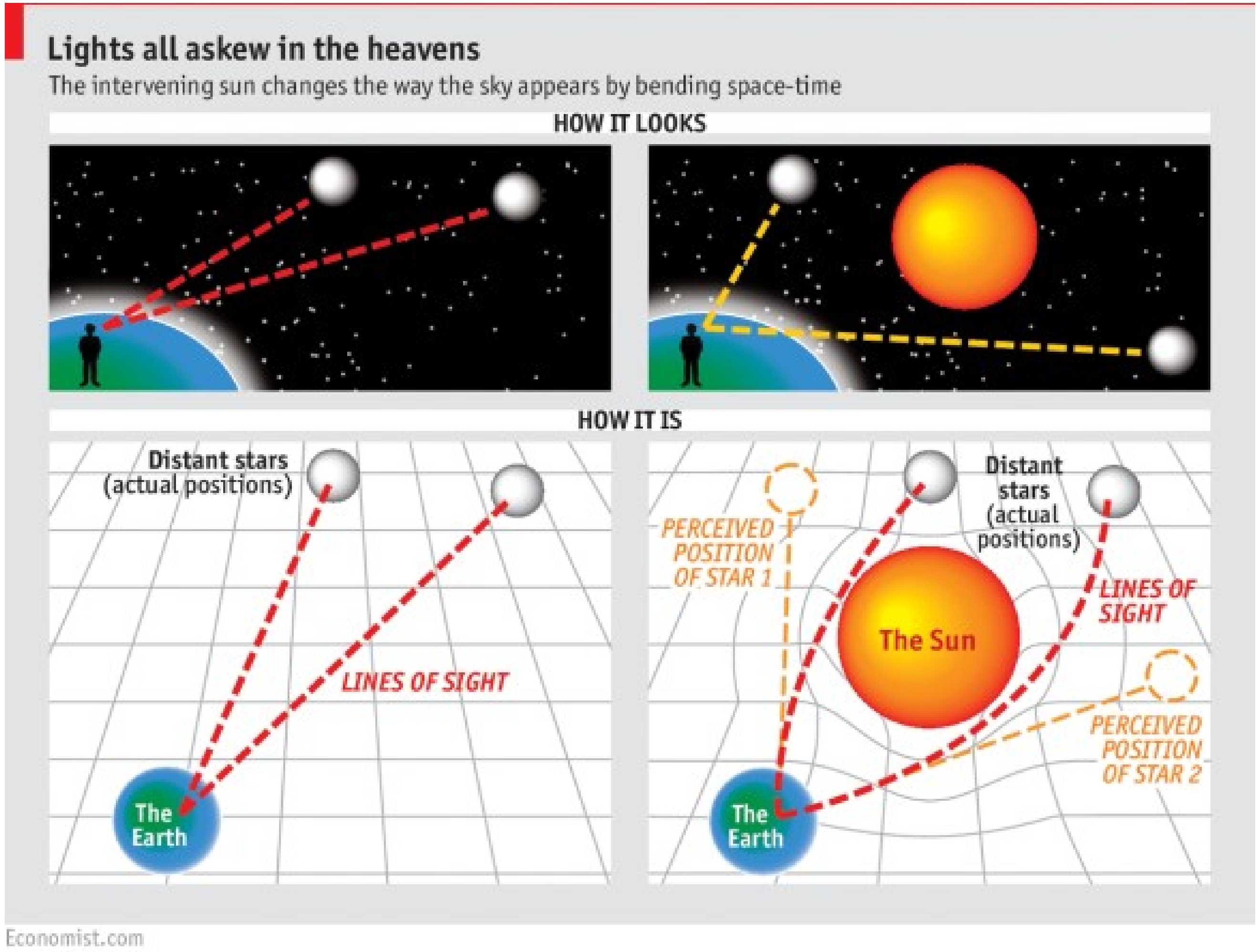
But in space the shortest possible path will always be a "straight" line, right? Right??

We know that light always takes the shortest path to the destination.

Light travels along the spacetime geodesic.

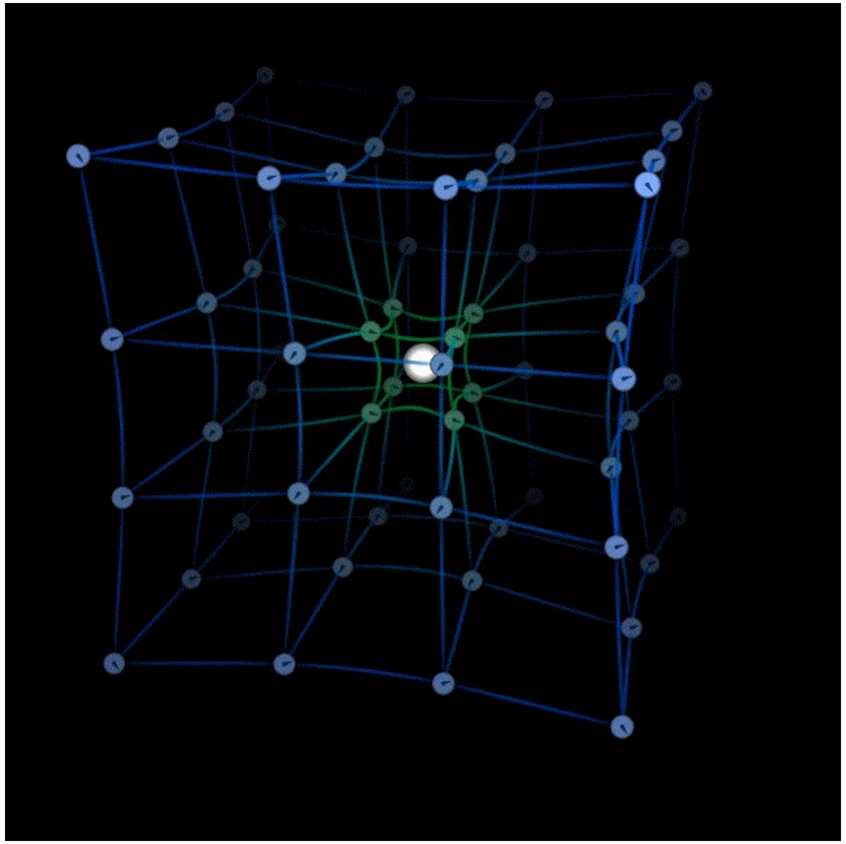
But what if I tell you that light does 'curve'?

According to the General Theory of Relativity, presence of matter and energy in space curves the spacetime around it, changing the angles and distances. While you follow this curved path, you will not know it but from afar it will look curved.

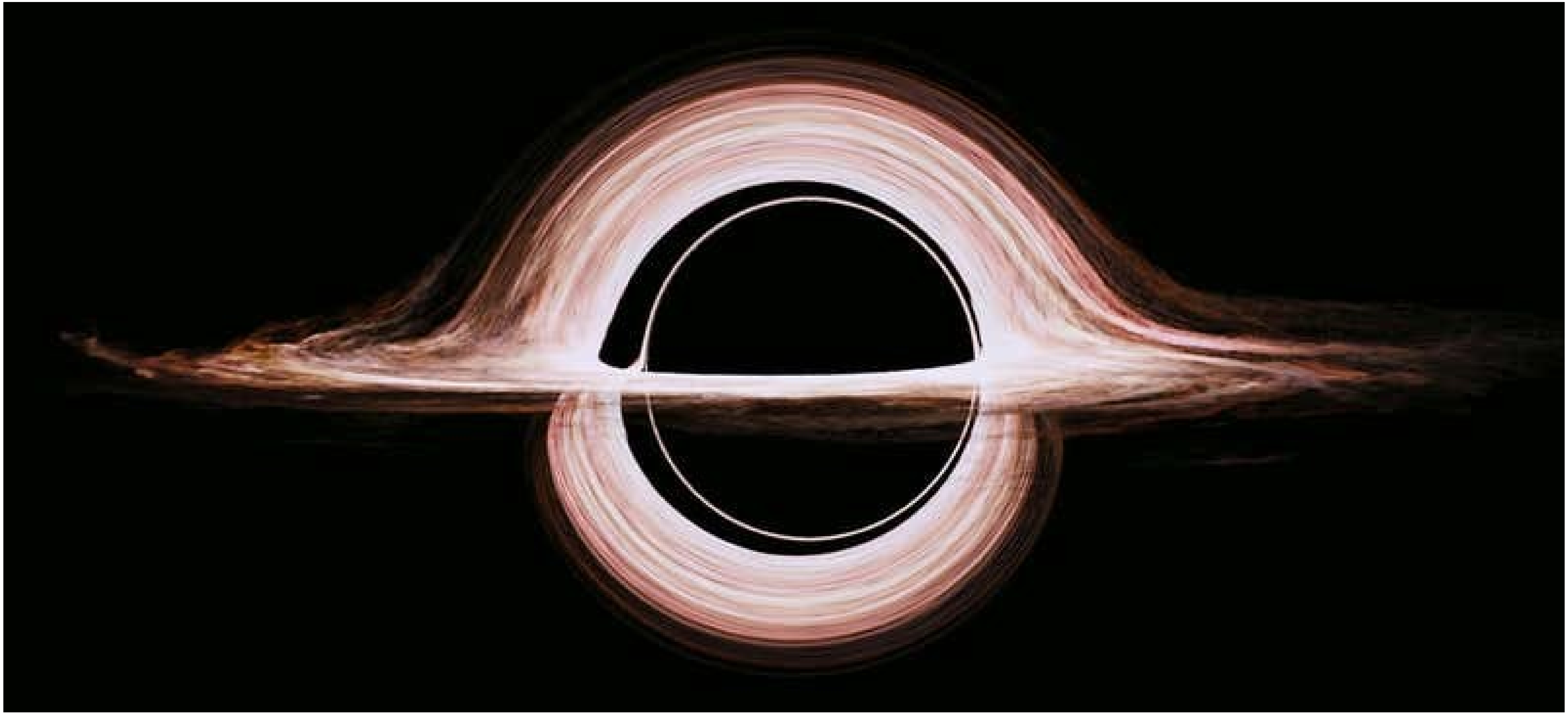


The metric tensor around a massive spherical object (like a star) (in spherical coordinates) is:

$$ds^2 = [dt\ dr\ d\theta\ d\phi]\begin{bmatrix} \left(1-\dfrac{2GM}{rc}\right) & 0 & 0 & 0 \\ 0 & -\left(1-\dfrac{2GM}{rc}\right)^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 sin^2\theta \end{bmatrix}\begin{bmatrix} dt \\ dr \\ d\theta \\ d\phi \end{bmatrix}$$

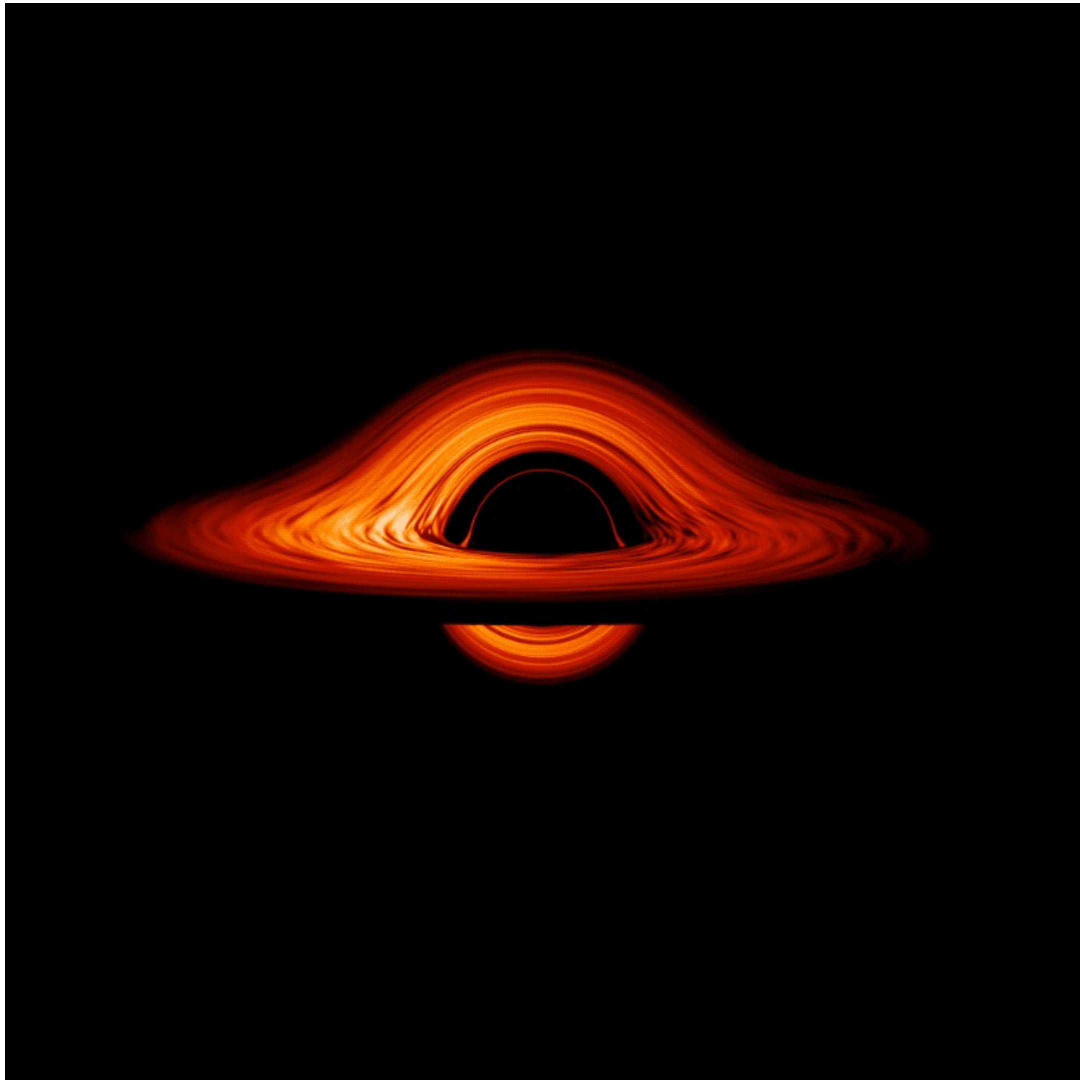


Remember this? This may seem like a weird shape but it is actually only a bright flat ring sitting horizontally around a spherical(!) object.



Here is a simulation of an accretion disk (think of a star but if it were a disk) around a black-hole rotating.

Because a black-hole bends light so much, you see the part of the disk behind the black-hole.



So, remember our formula to calculate the sum of the internal angles of a triangle?

$$\sum_{vertex1}^{vertex3} angle = \pi + \frac{A}{r^2}$$

Now we need to modify it to add the distortion caused by Gravity:

$$\sum_{vertex1}^{vertex2} angle = \pi + \frac{A^2}{r^2} * \left(\frac{GM_\oplus}{rc^2}\right)$$

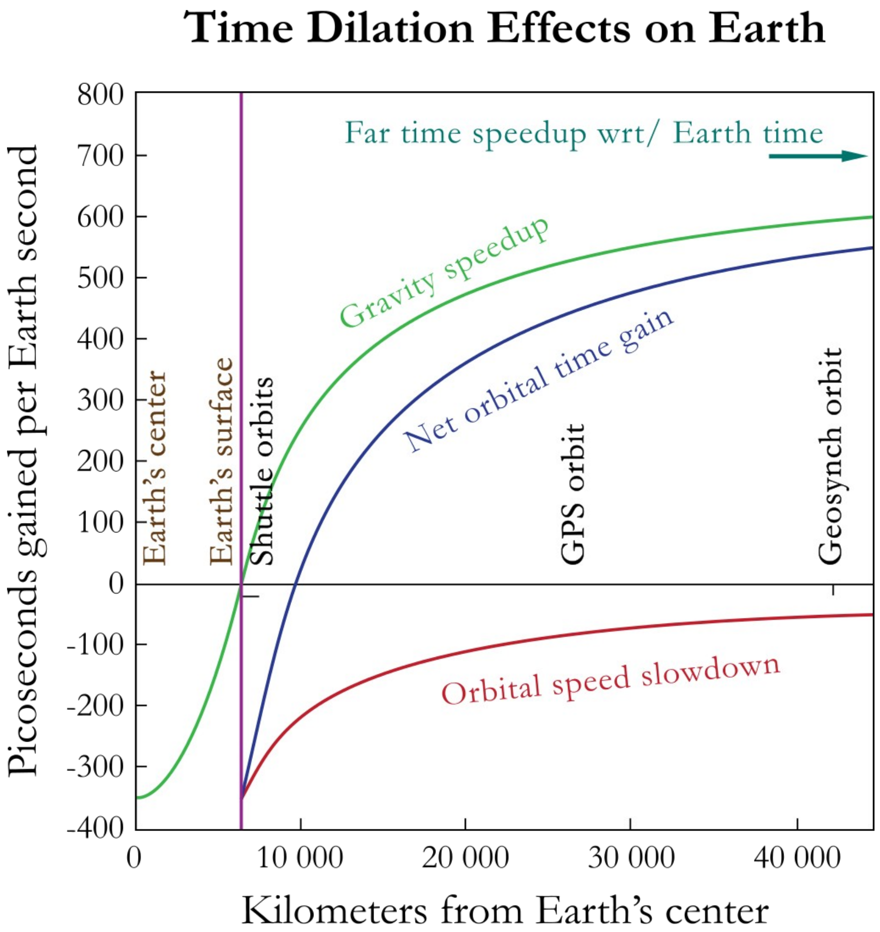
Gravity also causes time-dilation!

- The closer you are to a massive body, the slower your clock will tick as compared to someone who is far away.
- Here is how much:

$$t_0 = t_f \sqrt{1 - \frac{2GM}{rc^2}}$$

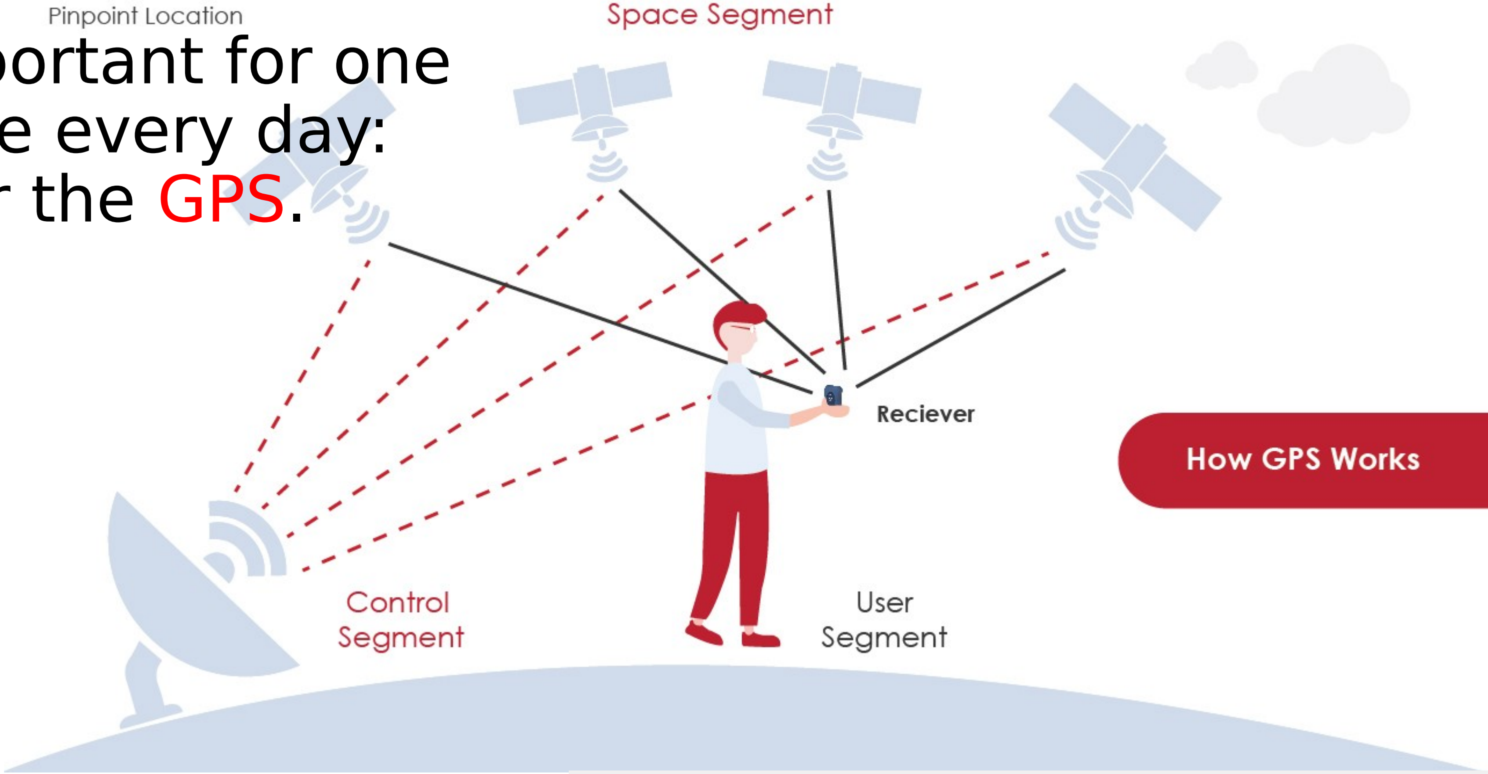
Here t0 is the time near the surface of a massive spherical body (the Earth for example), tf is the time very very far away and the remaining symbols have the usual meanings.

So that means that if an artificial satellite is orbiting the Earth, it experiences both time dilation due to the speed and time contraction due to lesser gravity.



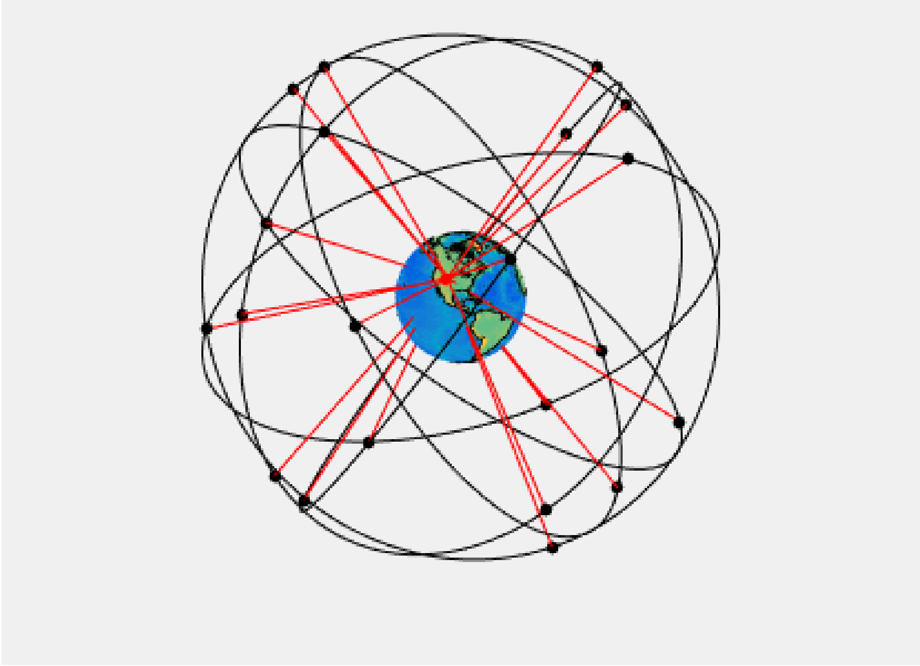
This precise effect, even though it is only a few hundred picoseconds per second is extremely important for one crucial thing that we use!

This calculation is extremely important for one particular technology that we use every day: The Global Positioning System or the GPS.



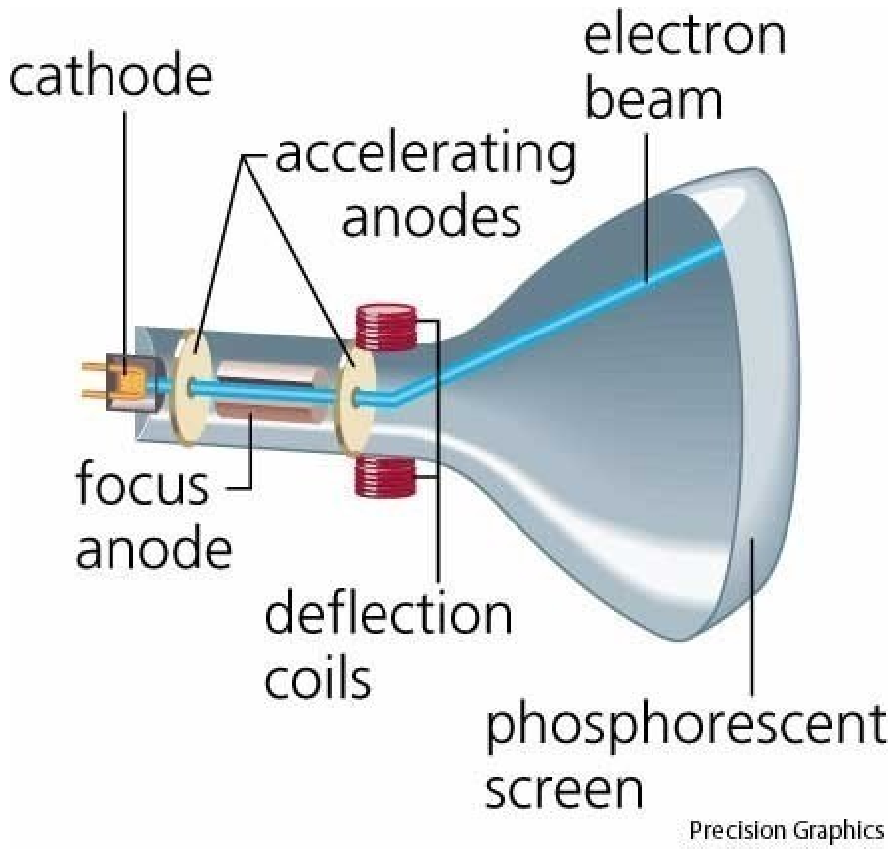
There are 31 GPS satellites in orbit around the Earth and at any time at least 4 of them are visible to your GPS tracking device. All of these satellites contain an atomic clock that ticks once a nanosecond. They all transmit signals at the same time.
Your position is calculated by the device by comparing the differences in arrival times of different signals.

Without the non-Euclidean conception of distances in the spacetime, we wouldn't have been able to calculate the gravitational time dilation!
Without that knowledge, the GPS satellites would have been faster than the Earth clocks by 37 microseconds each day!
GPS positioning would have been inaccurate in two minutes and off by 10km in a day!



Bonus:

Cathode Ray Tube TeleVision sets require relativistic equations to manipulate electrons travelling at super high speeds.

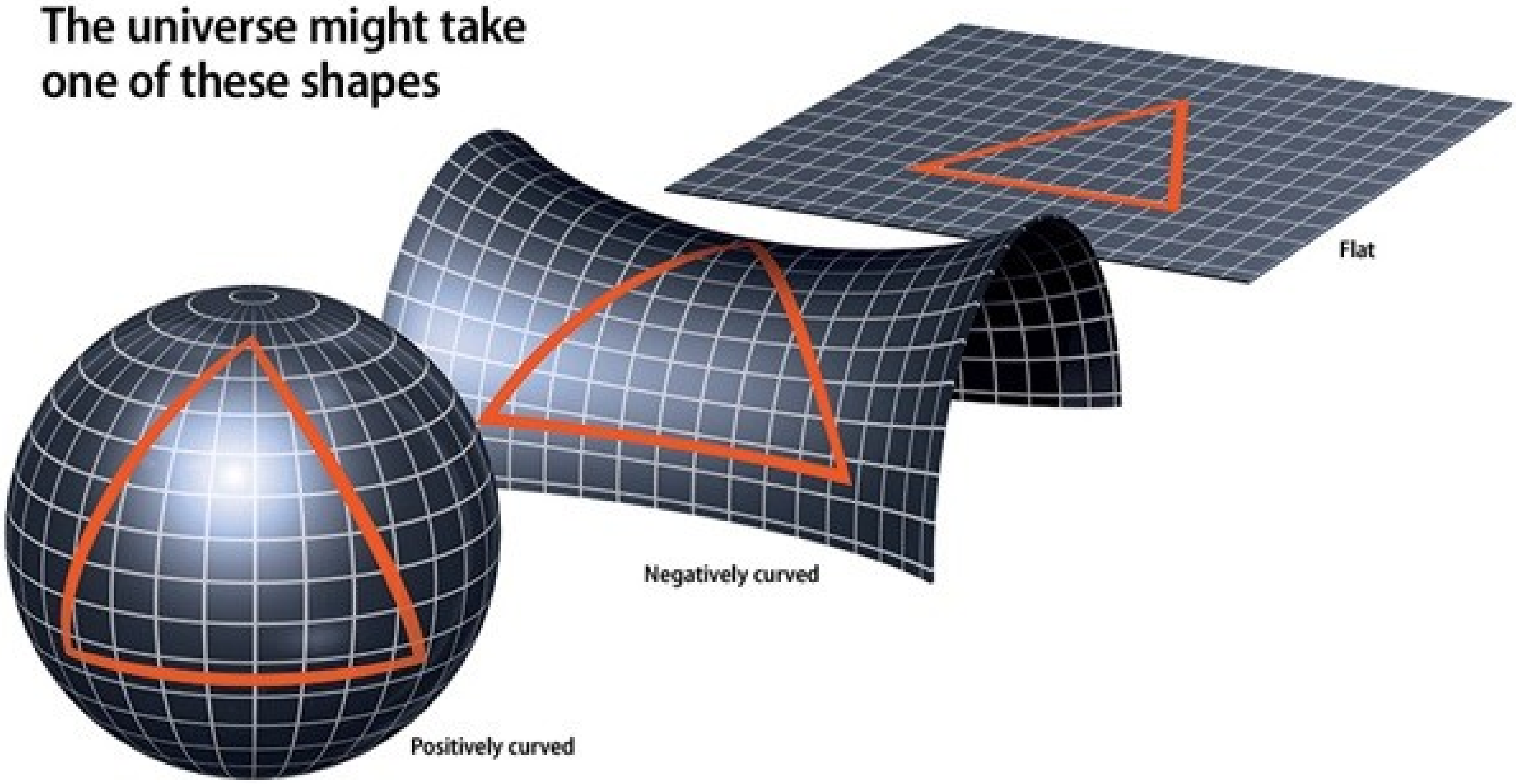


If you travelled back to the past when they did not know that the surface of the Earth was curved;

and informed them that if they keep walking or sailing in a straight line, they will end up where they started; they would think you are crazy!

What if I tell you that if you keep going in the same direction in outer space you might end up where you started?

It is certainly possible!
However, the current observations indicate that the universe is likely flat.



- The take-away from this talk is that our concepts of lines, planes, angles and distance are very local and small-scaled.
- We must appreciate the scale of our existence and the beauty
  of the incomprehensibly huge cosmos.
- What we think of as abstract or useless mathematics might tomorrow enable us to reach great heights!

Any

Questions?

Thank You

For Your Attention